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A THEORY OF INPUT–OUTPUT ARCHITECTURE

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Individual producers exhibit enormous heterogeneity in many dimensions. This paper develops a theory in which the network structure of production—who buys inputs from whom—forms endogenously. Entrepreneurs produce using labor and exactly one intermediate input; the key decision is which other entrepreneur's good to use as an input. Their choices collectively determine the economy's equilibrium input–output structure, generating large differences in size and shaping both individual and aggregate productivity. When the elasticity of output to intermediate inputs in production is high, star suppliers emerge endogenously. This raises aggregate productivity as, in equilibrium, more supply chains are routed through higher-productivity techniques.

KEYWORDS: Networks, productivity, supply chains, size distribution, input–output structure, economic growth, ideas.

> "The social payoff of an innovation can rarely be identified in isolation. The growing productivity of industrial economies is the complex outcome of large numbers of interlocking, mutually reinforcing technologies, the individual components of which are of very limited economic consequence by themselves. The smallest relevant unit of observation is seldom a single innovation but, more typically, an interrelated clustering of innovations."
>
> Rosenberg (1979, pp. 28–29)

ONE OF THE MORE STRIKING FEATURES of microeconomic production data is the enormous heterogeneity across producers. Researchers have documented vast differences in employment, sales, and, more recently, engagement in input–output linkages.[1] These observations have generated interest in two key questions: Why is there so much microeconomic heterogeneity? And what are the implications of this heterogeneity for aggregate productivity?

This paper develops a theory in which the economy's input–output architecture arises endogenously and shows that enormous differences in size can emerge even when differences in productivity are arbitrarily small. The theory offers a new perspective relative to canonical models of firm dynamics that typically explain dispersion in size as resulting from the accumulation of random growth of productivity or demand residuals.[2]

The theory is based on the premise that there may be multiple ways to produce a good, each with a different set of inputs. In the model, each entrepreneur sells a particular good

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[1]See, for example, Axtell (2001), Rossi-Hansberg and Wright (2007), Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012), and Atalay, Hortacsu, Roberts, and Syverson (2011).

[2]These include Gibrat (1931), Simon and Bonini (1958), Hopenhayn (1992), Klette and Kortum (2004), Rossi-Hansberg and Wright (2007), and Luttmer (2007).

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and can use a variety of techniques to produce that good. Each technique allows the entrepreneur to produce her good using labor and exactly one other entrepreneur's good as an intermediate input, with productivity specific to that input. The key decision is which other entrepreneur's good to use. An entrepreneur's cost of production when using one of these techniques depends both on the technique's productivity and on the price of the intermediate input which, in turn, depends on the production cost of the entrepreneur that produces that input. In this environment, the economy's production possibilities cannot be summarized in terms of the capabilities of individual producers. The collection of known production techniques forms a network comprising each entrepreneur's potential suppliers and potential customers—others who might use the entrepreneur's good as an intermediate input. When producing, each entrepreneur selects from her techniques the one that is most cost-effective. These choices collectively determine each producer's size and contribution to aggregate productivity.

Terms of trade determine which of the potential input–output linkages are used in equilibrium and the extent to which one producer's low cost of production is passed on to others. Producers engage in standard monopolistic competition in sales of their goods to a representative household for consumption, but bilateral contracts govern each pairwise transaction. I restrict attention to arrangements that are countably-stable; terms of trade must be such that countable coalitions cannot find alternative terms that are mutually beneficial.[3] In any such equilibrium, production within each supply chain is efficient.

The paper's main results describe how individual choices lead to the endogenous emergence of star suppliers and the implications of this for aggregate outcomes.[4] Star suppliers are individual entrepreneurs that, in equilibrium, sell their goods to many other entrepreneurs for intermediate use.[5] For an entrepreneur to be a star supplier, she must have many potential customers, and a large fraction of those potential customers must choose to use her good in equilibrium. Whether those potential customers choose her good depends both on the price she is willing to accept for her good and on how much those potential customers are willing to trade off a technique with a supplier that offers a low price for a technique with a high match-specific productivity.[6]

One striking feature of this environment is that the prevalence of star suppliers—and hence dispersion in size—is independent of the parameter that determines dispersion in entrepreneurs' marginal cost. Thus even with arbitrarily little variation in entrepreneurs' marginal costs, there can still be large differences in size.

To shed light on the economic forces shaping the organization of production and aggregate productivity, I impose a functional form assumption that allows for an analytical characterization of a number of features of the economy. The key part of the assumption

[3]I also discuss the implications of a weaker restriction, pairwise stability. Pairwise stable arrangements are not robust to deviations that would be natural in this environment. For example, if an entrepreneur agrees to terms with a new supplier that would lower her marginal cost, it would be natural for that entrepreneur to reoptimize terms of trade with any buyers (and for those buyers to reoptimize terms of trade with their buyers, etc.), and for the new supplier to alter the quantity of inputs she purchases from her supplier, etc.

[4]The model is constructed to speak to patterns of linkages among individual producers rather than broader patterns of which sectors buy inputs from which other sectors. See Carvalho and Voigtländer (2014) for a promising step in that direction.

[5]Building on Gabaix (2011), Acemoglu et al. (2012) showed that the prevalence of star suppliers is one factor that determines whether idiosyncratic shocks are relevant for aggregate fluctuations.

[6]Since Rosen (1981), the superstar literature studies how small differences in talent can lead to large differences in compensation. The key factor is the relationship between differences in talent and how much customers are willing to trade off differences in talent for a lower price. While the analogy is not perfect, similar forces are at play here.

is that the distribution from which techniques' productivities are drawn has a right tail that follows a power law with exponent ζ. Under this assumption, I derive four results about the way entrepreneurs' selections of suppliers interact to determine the size distribution, matching patterns, expenditure shares, and aggregate productivity.

I first identify features of the environment that determine how customers are distributed across entrepreneurs and the size distribution. The distribution of customers depends on a single parameter, α; each technique is a Cobb–Douglas production function, and α is the elasticity of output of the buyer's good with respect to the supplier's good. One measure of the prevalence of star suppliers is the thickness of the right tail of this distribution. I show that the distribution has a power-law tail with exponent $1/\alpha$. Why does α play the key role? Recall that when an entrepreneur selects a supplier, she considers both the match-specific productivity and the cost of the associated input of each of her techniques. When α is small, the cost of the inputs is less important, and thus suppliers' costs are less important drivers of choices of suppliers. Conversely, when α is large, entrepreneurs with low production costs are selected as suppliers more systematically, and are therefore more likely to be star suppliers. In line with the results mentioned above, the distribution does not depend on ζ, which indexes dispersion in marginal costs.

The distribution of customers across entrepreneurs is one determinant of the size distribution. An entrepreneur's size, as measured by employment, depends on her sales to the household and her sales to each customer, which depends on the sizes of those customers. It turns out that, under the functional form assumption, the size distribution has a threshold property. If α is small, the model has the property found in many other models that the right tail is dominated by those that sell large quantities to the household, and the shape of the right tail depends on dispersion of marginal cost (ζ). If α is larger, intermediate inputs become more important in production and sales of intermediate inputs become more concentrated in star suppliers. If α is large enough to cross a threshold, the right tail becomes dominated by star suppliers and follows a power law with exponent $1/\alpha$. Thus the extent of heterogeneity in productivity or in marginal cost no longer plays any role in determining the shape of the right tail.

Second, I show that one's conclusions about whether the equilibrium features assortative matching depends on which attribute one focuses on. Among equilibrium matches, buyers' and suppliers' marginal costs are uncorrelated, but their sizes (as measured by employment) are positively correlated. For a randomly selected technique, if the potential supplier has a low cost of production, the technique is likely to deliver a low cost of production to the potential buyer. In equilibrium, however, techniques are not selected randomly. An entrepreneur is likely to select a technique whose supplier has a low marginal cost even if the match-specific productivity is poor, but is unlikely to select a technique whose supplier has a high marginal cost unless the match-specific productivity is especially high. As a result, among matches observed in equilibrium, buyers' and suppliers' marginal costs are uncorrelated.[7] Despite this, their sizes are positively correlated; a supplier whose

[7] One of the most promising applications of data on firm-to-firm transactions is using the observed network structure to learn about the production process—who supplies inputs to whom and how these interactions change over time. Since much of the literature is focused on the source of fluctuations, an effort has been made to identify shocks and decompose the fluctuations (Foerster, Sarte, and Watson (2011), Atalay (2017), Di Giovanni, Levchenko, and Méjean (2014), Kramarz, Martin, and Mejean (2016)). An important lesson from the labor literature (Abowd, Kramarz, and Margolis (1999), Lopes de Melo (2018)) is that ignoring the endogeneity of match formation can bias inference about individual characteristics. The fact that individuals select into a match provides information about those individuals and about the quality of the match. The same biases arise in the environment presented here.

customer is unusually large needs to hire an unusually large amount of labor to produce the intermediates for that customer.

Third, while dispersion of marginal cost has no impact on the prevalence of star suppliers, it does determine the aggregate cost share of intermediate inputs. In a competitive equilibrium, the cost share of intermediate inputs would be α. Here, however, since buyers and suppliers split surplus, the cost share is weakly larger than α.[8] While countable stability does not pin down how producers split surplus, there is a class of equilibria indexed by a parameter that has a natural interpretation as bargaining power.[9] Notably, within this class, the aggregate cost share depends on ζ, which indexes dispersion in marginal cost and is related to the size of the loss when an entrepreneur must switch to its next best supply chain.

Finally, the emergence of star suppliers affects aggregate productivity in the following sense: entrepreneurs' selections jointly determine the supply chains used to produce each good. Aggregate productivity depends on the match-specific productivity of the techniques used at each step in each of those supply chains. When α is larger, supply chains are more likely to be routed through the most productive techniques in the economy, raising aggregate productivity. This channel complements the usual input–output multiplier that is present in all models with roundabout production.[10]

Related Literature

The model provides a mechanism that delivers a skewed cross-sectional distribution of links that complements but differs from that of the influential preferential attachment model of Barabasi and Albert (1999).[11] Here, even though the distribution of potential customers follows a distribution with a thin tail, the distribution of actual customers endogenously exhibits more variation. The mechanism is a natural one once choices are endogenous: entrepreneurs that are willing to accept a lower price for their goods are systematically more likely to be selected as suppliers by their potential customers.

Similarly, if α is large enough, the model's predictions for the size distribution differ substantively from canonical models of producer dynamics referenced in footnote 2. In those models, dispersion in producer size results from the accumulation of idiosyncratic

[8]Thus in contrast to environments such as Hulten (1978) in which producers are price takers, the cost share of intermediate inputs is not a sufficient statistic for the input–output multiplier.

[9]The bargaining power parameter is analogous to that of the generalized Nash bargaining solution. The difference here is that the size of surplus to be split between a supplier and buyer depends on how the buyer splits surplus with *her* buyers, etc.

[10]By now it is well understood that input–output linkages play a key role in shaping how one producer's cost of production impacts other producers and ultimately aggregate productivity. The literature on aggregate fluctuations has focused on how sectoral shocks propagate through an economy—see Long and Plosser (1983), Horvath (1998), Dupor (1999), Acemoglu et al. (2012), and Carvalho and Gabaix (2013). In the growth literature, see Jones (2011, 2013) and Ciccone (2002). In the trade literature, Caliendo and Parro (2015) argued that the welfare impact of changes in tariffs in particular industries depends on the input–output structure.

[11]The preferential attachment model was designed because network models with uniform arrival of links exhibit link distributions with thin tails, whereas many real-world networks feature link distributions with power-law tails. In the preferential attachment model, there is an initial network and, over time, new links are formed and new nodes are born. The probability that a new link involves a particular node is increasing in the number of links that node already has. Variants of the preferential attachment model of Barabasi and Albert (1999) have been used to explain the cross-sectional distribution of the firm-to-firm linkages (Atalay et al. (2011)) and the network structure of international shipments of goods (Chaney (2014)). See also Kelly, Lustig, and Van Nieuwerburgh (2013). The mechanism is similar to proportional random growth models and goes back to Yule (1925) and Simon (1955).

productivity or demand shocks. The model here is simple in the sense that there is a single type of shock: the arrival of a technique. But there are two key differences from productivity shocks of canonical models. First, a technique represents both a way to produce one good but also a way of using a different good.[12] In other words, the arrival of a technique is a supply shock to one producer (the buyer) but a demand shock to another producer (the supplier). Second, a producer is affected by shocks to others (i.e., to customers' customers or to suppliers' suppliers, etc.). It is the interaction of upstream and downstream techniques and the higher-order interconnections that generates the rich cross-sectional patterns.

The model is written in such a way that it nests a simple version of Kortum (1997) when α is zero—a special case in which the network structure plays no role.[13] This special case provides a backdrop against which one can see the network structure's role in shaping economic outcomes.

Finally, the market structure builds on Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp (2013) who studied stable outcomes in supply chain networks. The main substantive difference is that I focus on countably-stable outcomes and splits of surplus consistent with bargaining power, whereas Hatfield et al. (2013) focused on fully stable outcomes and compared them to competitive equilibria. The main technical difference is that the environment studied here contains a continuum of agents, which allows the use of the law of large numbers but raises some technical issues such as defining feasible outcomes.[14]

1. THE ENVIRONMENT

There is a unit mass of entrepreneurs. Each is the sole producer of a differentiated good. A representative household derives utility from consuming all of the goods, consuming the bundle $C = (\int_0^1 c_j^{\frac{\varepsilon-1}{\varepsilon}} \, dj)^{\frac{\varepsilon}{\varepsilon-1}}$ where c_j is consumption of good j. The household supplies L units of labor inelastically.

To produce, an entrepreneur must adopt a technique. A technique allows an entrepreneur to produce her good using labor and exactly one other entrepreneur's good as an intermediate input. We denote a single technique by ϕ and the set of all techniques by Φ. A technique ϕ involves a particular buyer, denoted $b(\phi)$, and a particular supplier, $s(\phi)$, and has a match-specific productivity $z(\phi)$. A technique is fully defined by these three characteristics. The technique describes the production function

$$y = \frac{1}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} z x^{\alpha} l^{1-\alpha}, \tag{1}$$

where y units of the buyer's good are generated using x units of the supplier's good and l units of labor.

The economy's production possibilities can be completely described by the set of all techniques available to the different entrepreneurs, Φ. Techniques arrive randomly via a

[12]The idea that innovation is finding a new use for an existing good is related to recombinant growth of Weitzman (1998), in which innovation is finding a better way to combine inputs. Scherer (1982) and Pavitt (1984) showed that roughly three quarters of innovations are for use by others outside the innovating firm's sector.

[13]The model is also related to Alvarez, Buera, and Lucas Jr. (2008), Lucas Jr. (2009), Lucas Jr. and Moll (2014), and Perla and Tonetti (2014). See Buera and Oberfield (2016) for an elaboration of how the mathematical structure of this model is related to models of idea flows.

[14]Countable-stability is also related to f-core of Hammond, Kaneko, and Wooders (1989) who studied deviations of finite coalitions in a continuum economy.

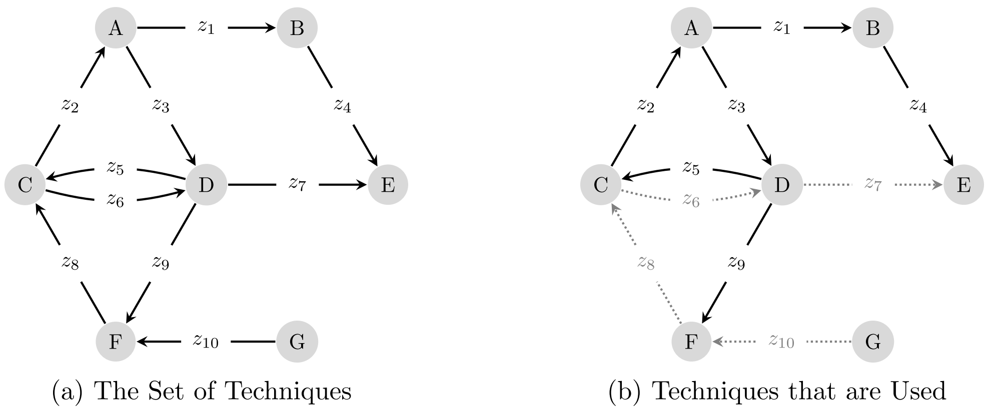

FIGURE 1.—A graphical representation of the input–output structure. Panel (a) gives an example of a set of techniques, Φ. Each node is an entrepreneur and edges correspond to techniques. An edge's direction indicates which entrepreneur produces the output and which provides the input. An edge's number represents the productivity of the technique. In panel (b), solid arrows are techniques that are used in equilibrium.

process described below. Any realization of the set of techniques can be represented by a weighted, directed graph. Figure 1(a) gives an example of a realization for an economy with a finite number of entrepreneurs. In the figure, each entrepreneur is a node and each technique is an edge connecting two nodes. Each edge has a direction, indicating which entrepreneur would supply the intermediate input and which would produce the output, and a number corresponding to the technique's match-specific productivity.

Some entrepreneurs may have multiple ways to produce whereas others may have none. For the realization of Φ depicted in Figure 1(a), it is infeasible for G to produce, but E has a technique to produce using good D as an input and another that uses good B. In principle, she may produce using both, although it will be shown later that, in equilibrium, she will generically produce using only one. While Figure 1(a) completely describes the economy's production possibilities, Figure 1(b) gives an example of entrepreneurs' selections of which techniques to use. Such selections, which ultimately depend on prices, correspond to vertical relationships we would observe in equilibrium and jointly determine the supply chains that are used to produce each good.

The set of techniques Φ is a random set. The number of techniques with which an entrepreneur can produce her good follows a Poisson distribution with mean M. For each of those techniques, the identity of the supplier $s(\phi)$ is random and uniformly drawn from all entrepreneurs in the economy. This incidentally implies that the number of techniques for which an entrepreneur is the supplier also follows a Poisson distribution with mean M. Each technique's match-specific productivity, $z(\phi)$, is drawn from a fixed distribution with CDF H.[15] Assumption 1 guarantees that the right tail of this distribution is not too thick, which is sufficient to ensure that aggregate output is almost surely finite.

ASSUMPTION 1: *The support of H is bounded below by some $z_0 > 0$ and there exists a $\beta > \varepsilon - 1$ such that* $\lim_{z\to\infty} z^{\beta}[1 - H(z)] = 0$.

[15]To keep the notation manageable, I abstract from any ex ante differences in goods' suitability for use as an intermediate input for any other type of good or for consumption. In many applications, ex ante asymmetries across goods would be important (reflecting different industries or countries). The model can easily accommodate such asymmetries without loss of tractability; see working paper version.

Φ implicitly determines the supply chains that may be used to produce each good. For an entrepreneur, a supply chain consists of a technique she can use to produce her good, a technique that the associated supplier can use, etc. Formally, a *supply chain* for entrepreneur j is an infinite sequence of techniques $\{\phi_k\}_{k=0}^{\infty}$ with the property that $j = b(\phi_0)$ and $s(\phi_k) = b(\phi_{k+1})$ for each k. Supply chains may cycle, in which case the techniques in the sequence repeat.[16] It is feasible for an entrepreneur to produce only if there exists at least one supply chain to produce that entrepreneur's good.[17,18]

2. MARKET STRUCTURE

Terms of trade among entrepreneurs determine entrepreneurs' choices of inputs, production decisions, and productivity. This section describes a market structure and the terms of trade that arise in equilibrium. The market structure is motivated by the idea that groups of entrepreneurs that are small (relative to the market) may choose terms that maximize their mutual gains from trade.

Entrepreneurs engage in monopolistic competition when selling to the representative household, but sales of goods for intermediate use are governed by bilateral trading contracts specifying a buyer, a supplier, a quantity of the supplier's good to be sold to the buyer, and a payment. Given a contracting arrangement, each entrepreneur makes her remaining production decisions to maximize profit. Goods may not be resold and entrepreneurs remit all profit to the household.

The economy is in equilibrium when the arrangement is such that no coalition of entrepreneurs (of a specified size) would find it mutually beneficial to deviate by altering terms of trade among members of the coalition and/or dropping contracts with those not in the coalition.

2.1. *Arrangements and Payoffs*

Terms of trade are described by a contracting arrangement. A *contract* for technique ϕ is a double $\{x(\phi), T(\phi)\}$, where $x(\phi)$ is the quantity of good $s(\phi)$ to be sold to the buyer $b(\phi)$, and $T(\phi)$ is a payment from the buyer to the supplier. An *arrangement* is a contract for each technique, $\{x(\phi), T(\phi)\}_{\phi\in\Phi}$.[19] For some techniques, the arrangement might specify that $x(\phi) = T(\phi) = 0$, which means that the buyer does not use those techniques.

Taking as given the arrangement, the wage, and the household's demand for its good, each entrepreneur chooses a price to charge the household for its good and how much

[16]For example, lumber is useful in the production of an ax, while an ax is useful in the production of lumber.

[17]In the example with a finite set of entrepreneurs, any supply chain must contain a cycle. With a continuum of entrepreneurs, supply chains need not cycle and the set of entrepreneurs who have supply chains that cycle has measure zero.

[18]The assumptions that production of each good requires the use of some other good as an input and that chains of these relationships may continue indefinitely follow in the tradition of Leontief (1951) and other models with roundabout production.

[19]The notation rules out the possibility of a contract between a buyer and supplier when the buyer does not have a technique that uses the supplier's good as an input. Supplemental Material Appendix A.1 (Oberfield (2018)) shows that this is without loss of generality. Because goods may not be resold, such a transaction can never be optimal; a buyer would drop any contract with a positive payment whereas a supplier would drop all other contracts. Supplemental Material Appendix A.1 shows that such transactions are also not essential off the equilibrium path. The notation also rules out multiple contracts for the same technique. Supplemental Material Appendix A.1 shows that this restriction can be made without loss of generality as well.

labor to hire and allocate to each technique. If entrepreneur j uses $l(\phi)$ units of labor with technique ϕ, then its output of good j produced using that technique is $y(\phi) \equiv \frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}} z(\phi)x(\phi)^\alpha l(\phi)^{1-\alpha}$, following (1).

For entrepreneur j, there are two subsets of Φ that are directly relevant. Let $U_j \equiv \{\phi \in \Phi : b(\phi) = j\}$ be the set of techniques that designate j as the buyer (those upstream from j) and let $D_j \equiv \{\phi \in \Phi : s(\phi) = j\}$ be the set of techniques that designate j as the supplier (those downstream from j). These respectively determine j's potential suppliers and potential customers.

Total output of good j and total labor used by j are thus $y_j \equiv \sum_{\phi\in U_j} y(\phi)$ and $l_j \equiv \sum_{\phi\in U_j} l(\phi)$, respectively. This output can be sold to the household for consumption or to other entrepreneurs for intermediate use, which together sum to $c_j + \sum_{\phi\in D_j} x(\phi)$.

Given the arrangement, entrepreneur j's payoff is

$$\Pi_j = \max_{p_j, c_j, \{l(\phi)\}_{\phi\in U_j}} p_j c_j + \sum_{\phi\in D_j} T(\phi) - \sum_{\phi\in U_j} \big[T(\phi) + wl(\phi)\big] \tag{2}$$

subject to satisfying the household's demand

$$c_j \leq C(p_j/P)^{-\varepsilon}, \qquad P \equiv \left(\int_0^1 p_j^{1-\varepsilon}\, dj\right)^{\frac{1}{1-\varepsilon}}$$

and a technological constraint that total usage of good j cannot exceed total production of good j:

$$c_j + \sum_{\phi\in D_j} x(\phi) \leq \sum_{\phi\in U_j} \frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}} z(\phi)x(\phi)^\alpha l(\phi)^{1-\alpha}. \tag{3}$$

2.2. *Feasibility*

An allocation consists of quantities of labor and the intermediate input used for and output derived from each technique $\{l(\phi), x(\phi), y(\phi)\}_{\phi\in\Phi}$, and the household's consumption vector, $\{c_j\}_{j\in[0,1]}$. An allocation is *feasible* if it is both *resource feasible* and *chain feasible*. An allocation is *resource feasible* if it satisfies (3) for each j and the labor resource constraint,

$$\int_0^1 \left(\sum_{\phi\in U_j} l(\phi)\right) dj \leq L.$$

We impose a second requirement, chain feasibility, which is analogous to a no-Ponzi condition. A technique is typically part of many supply chains. While the allocation specifies the production using each technique, this production can be divided into separate pieces for use in the various supply chains that pass through that technique. Loosely, chain feasibility ensures that the consumption generated by each supply chain is feasible given the inputs used at each step in the supply chain.

Defining chain feasibility requires an alternative representation of the allocation that we label a supply chain representation. Formally, for entrepreneur j, let Ω_j denote the set of supply chains available to produce good j. Recall that each supply chain $\omega \in \Omega_j$ is a sequence of techniques $\{\phi_n\}_{n=0}^\infty$ with the properties that j is the buyer of the most downstream technique, that is, $j = b(\phi_0)$, and that the supplier of each technique is the buyer for the technique one step further upstream, $s(\phi_n) = b(\phi_{n+1})$ (or equivalently $\phi_{n+1} \in U_{s(\phi_n)}$). I will refer to the technique that is furthest downstream as technique 0 in the chain and the technique that is n stages upstream from technique 0 as technique n; a higher number indicates a technique that is further upstream.

A supply chain representation consists of $\{c(\omega), \{y^n(\omega), x^n(\omega), l^n(\omega)\}_{n=0}^{\infty}\}_{\omega\in\Omega_j, j\in[0,1]}$. For each chain $\omega \in \Omega_j$, $c(\omega)$ is total production of good j for consumption using supply chain ω, and for each $n \in \{0, 1, 2, \dots\}$, $\{y^n(\omega), x^n(\omega), l^n(\omega)\}$ are the quantities of output, intermediate input, and labor used with technique n in the supply chain ω in the production of $c(\omega)$.[20] A supply chain representation must satisfy four properties. First, the output in the final stage exactly matches what is consumed, $y^0(\omega) = c(\omega)$. Second, the intermediate input used at one stage exhausts the output from the previous next stage, $x^n(\omega) = y^{n+1}(\omega)$. Third, the inputs and output at each step in each chain are consistent with the technology of the technique used at that stage:

$$y^n(\omega) = \frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}} z^n(\omega) x^n(\omega)^\alpha l^n(\omega)^{1-\alpha}, \tag{4}$$

where $z^n(\omega)$ is the match-specific productivity of technique n of the supply chain. Fourth, the supply chain representation must actually generate the specified allocation.[21] Supplemental Material Appendix A.2 provides a more detailed treatment.[22]

An allocation is *chain feasible* if there exists a supply chain representation such that, for every chain,

$$c(\omega) \leq \liminf_{k\to\infty} \prod_{n=0}^{k} \left(\frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}} z^n(\omega) l^n(\omega)^{1-\alpha} \right)^{\alpha^n}.$$

To unpack this definition, note that combining (4) for stages $n = 0, \dots, k$ for a single chain ω and using $x^n(\omega) = y^{n+1}(\omega)$ yields $c(\omega) = x^k(\omega)^{\alpha^{k+1}} \prod_{n=0}^{k} (\frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}} z^n(\omega) l^n(\omega)^{1-\alpha})^{\alpha^n}$. Chain feasibility for ω imposes that the sequence of intermediate inputs used further and further upstream in the supply chain ω does not explode, that is, $\limsup_{k\to\infty} [x^k(\omega)]^{\alpha^{k+1}} = 1$.

2.3. *Stability and Equilibrium*

An allocation is an equilibrium when no coalition of entrepreneurs (of a specified size) can find alternative terms of trade that would be mutually beneficial. A coalition is a

[20] Supply chains overlap. To avoid double counting, the decomposition specifies production of supply chain $\omega \in \Omega_j$ as the production of good j for consumption rather than total output of good j, as the latter includes production of good j for intermediate use in other supply chains. Since every technique exhibits constant returns to scale, the decomposition is well-defined.

[21] For each technique, the supply chain representation defines production using that technique for each chain that passes through the technique, while the allocation defines *total* production using that technique. For the supply chain representation to generate the specified allocation, it must be that the latter is equal to the sum of the former, summing over all supply chains that pass through the technique. Formally, if $\Omega(n, \phi)$ is the set of chains in which ϕ is technique n, then $l(\phi) = \sum_{n=0}^{\infty} \sum_{\omega\in\Omega(n,\phi)} l^n(\omega)$, $x(\phi) = \sum_{n=0}^{\infty} \sum_{\omega\in\Omega(n,\phi)} x^n(\omega)$, and $y(\phi) = \sum_{n=0}^{\infty} \sum_{\omega\in\Omega(n,\phi)} y^n(\omega)$. Similarly, it must be that consumption of each good equals the total consumption produced by all supply chains for the good, $c_j = \sum_{\omega\in\Omega_j} c(\omega)$.

[22] If the allocation is such that each entrepreneur produces using a single technique, there is a unique supply chain representation; in instances in which at least one entrepreneur produces using multiple techniques, there are multiple supply chain representations. Nevertheless, for any supply chain representation, there is a unique allocation.

set of entrepreneurs, J. For a coalition, a *deviation* consists of (i) dropping any subset of contracts that involve at least one entrepreneur in J—setting both the quantity and the payment specified in the contract to zero; and (ii) altering the terms of any contract involving a buyer and a supplier that are both members of the coalition.

A *dominating deviation* for a given coalition is a deviation that would deliver weakly higher payoffs to all members of that coalition and strictly higher payoffs to at least one member of that coalition.

An arrangement is *N-stable* if there are no coalitions of size/cardinality N or smaller with dominating deviations. The main focus of the paper will be on countably-stable arrangements—arrangements for which no countable coalition has a dominating deviation, but the text below also discusses the restrictions imposed by pairwise stability ($N = 2$). Of course, every countably-stable arrangement is pairwise stable, but not vice versa.

So that payoffs are well-defined out of equilibrium, we must account for an arrangement that dictates that an entrepreneur is obliged to produce ($\sum_{\phi \in D_j} x(\phi) > 0$) but receives no intermediate inputs with which to produce ($\sum_{\phi \in U_j} x(\phi) = 0$). To this end, I adopt the convention that, in such a situation, the entrepreneur can meet its obligation at infinite cost.

Finally, we define an N-stable equilibrium. An *N-stable equilibrium* is an arrangement $\{x(\phi), T(\phi)\}_{\phi\in\Phi}$, entrepreneurs' choices, $\{p_j, c_j, \{l(\phi)\}_{\phi\in U_j}\}_{j\in[0,1]}$, and a wage w such that (i) given the wage, total profit, and prices, the consumption choices $\{c_j\}_{j\in[0,1]}$ maximize the representative household's utility; (ii) for each $j \in [0, 1]$, $\{p_j, c_j, \{l(\phi)\}_{\phi\in U_j}\}$ maximize j's payoff given the arrangement, the wage, and the household's demand; (iii) labor and final goods markets clear; (iv) there are no dominating deviations available to any coalition of size/cardinality N; (v) the equilibrium allocation is feasible.

2.4. *Properties of Equilibria*

Let λ_j be entrepreneur j's marginal cost, the multiplier on (3).[23] For any arrangement, it is optimal for j to sell her good to the household at the usual markup over marginal cost, $p_j = \frac{\varepsilon}{\varepsilon-1}\lambda_j$, and to hire labor so that her wage bill for using technique $\phi \in U_j$ is $wl(\phi) = (1-\alpha)\lambda_j y(\phi)$.

It will be convenient to define $q_j \equiv \frac{w}{\lambda_j}$ to be the inverse of j's marginal cost in units of labor. q_j is the efficiency with which good j can be produced, or for shorthand, j's *efficiency*. Let $Q \equiv (\int_0^1 q_j^{\varepsilon-1}\,dj)^{\frac{1}{\varepsilon-1}}$ be the usual Dixit–Stiglitz productivity aggregator of entrepreneurs' efficiencies. Proposition 1 characterizes the implications of both pairwise stability and countable stability and shows that Q is a measure of aggregate productivity for the economy.

PROPOSITION 1: *In any pairwise-stable equilibrium, entrepreneur j sells $c_j = q_j^{\varepsilon} Q^{1-\varepsilon} L$ units of her good to the household for consumption, the household's consumption aggregate is*

$$C = QL, \tag{5}$$

[23]Note that j's marginal cost would be infinite, that is, $\lambda_j = \infty$, if either U_j is empty or j has no intermediate inputs (i.e., $\sum_{\phi\in U_j} x(\phi) = 0$).

and entrepreneurs' efficiencies satisfy

$$q_j = \max_{\phi\in U_j} z(\phi) q^{\alpha}_{s(\phi)}, \quad \textit{or} \quad q_j = 0 \quad \textit{if } U_j \textit{ is empty}, \tag{6}$$

$$q_j \leq \sup_{\omega\in\Omega_j} \prod_{n=0}^{\infty} \left[z^n(\omega)\right]^{\alpha^n}, \quad \textit{or} \quad q_j = 0 \quad \textit{if } \Omega_j \textit{ is empty}. \tag{7}$$

In any countably-stable equilibrium,

$$q_j = \sup_{\omega\in\Omega_j} \prod_{n=0}^{\infty} \left[z^n(\omega)\right]^{\alpha^n}, \quad \textit{or} \quad q_j = 0 \quad \textit{if } \Omega_j \textit{ is empty}. \tag{8}$$

Every countably-stable equilibrium is efficient. *If H is atomless*, *then with probability* 1, *there is an essentially unique allocation consistent with countable stability*.

Equation (6) summarizes how pairwise stability connects each entrepreneur's marginal cost to that of its suppliers. The proof shows that, for every technique,

$$\lambda_{b(\phi)} \leq \frac{1}{z(\phi)} \lambda^{\alpha}_{s(\phi)} w^{1-\alpha}, \quad \text{with equality if } x(\phi) > 0. \tag{9}$$

To understand this, note that for any technique, the buyer's marginal cost of output from using that technique is $\frac{1}{z(\phi)}\chi^{\alpha}_{s(\phi)}w^{1-\alpha}$, where $\chi_{s(\phi)}$ is the shadow value the buyer places on the supplier's good. The key implication of pairwise stability is that, for each technique that is actually used (i.e., $x(\phi) > 0$), the buyer's shadow value of an input must equal the supplier's marginal cost, as this maximizes the bilateral gains from trade.[24] For entrepreneur j, collecting these conditions for all techniques in U_j and using $q_j = w/\lambda_j$ gives (6).

Along similar lines, (5) says that in any pairwise stable equilibrium, aggregate productivity is simply the usual Dixit–Stiglitz productivity aggregator of entrepreneurs' efficiencies. The key step in the proof is to show that pairwise stability implies that each entrepreneur's marginal cost—measured in units of labor—is equal to the quantity of labor used across all stages in all supply chains to produce a unit of that entrepreneur's good. In other words, pairwise stability is sufficient to ensure that no supply chain suffers from double marginalization; each pair maximizes their bilateral gains from trade, which ensures that the quantity in the contract is such that the supplier's marginal cost is equal to the buyer's shadow value of the input.

Equation (7) is an implication of chain feasibility. For a single chain, $\omega \in \Omega_j$, this inequality follows from combining (6) for entrepreneur j, for its suppliers, for its suppliers' suppliers, etc. and imposing chain feasibility.[25]

Equation (8) shows that if an arrangement is countably-stable, entrepreneur j's efficiency equals that of its most efficient supply chain.[26] The proof shows that, if there is

[24]If $\lambda_{b(\phi)} < \frac{1}{z(\phi)}\lambda^{\alpha}_{s(\phi)}w^{1-\alpha}$, the buyer and supplier could increase their bilateral surplus by lowering $x(\phi)$. Such a deviation is always possible unless $x(\phi) = 0$. If $\lambda_{b(\phi)} > \frac{1}{z(\phi)}\lambda^{\alpha}_{s(\phi)}w^{1-\alpha}$, the buyer and supplier could increase their bilateral surplus by raising $x(\phi)$.

[25]Chain feasibility rules out solutions to (9) in which $\lambda_{s(\phi)} = \lambda_{b(\phi)} = 0$. While there is always such a solution to (9), there is no feasible allocation of labor that would deliver the supplier's good at a marginal cost of zero.

[26]The restrictions placed by pairwise stability on entrepreneurs' efficiencies are relatively weak. One clear example: For any Φ, there is always a pairwise stable equilibrium in which no entrepreneur produces, with

an entrepreneur j with $q_j < \sup_{\omega\in\Omega_j}\prod_{n=0}^{\infty}[z^n(\omega)]^{\alpha^n}$, then there is a countable coalition comprising entrepreneurs that form some chain $\omega\in\Omega_j$ with $\prod_{n=0}^{\infty}[z^n(\omega)]^{\alpha^n} > q_j$ with a dominating deviation in which each of those entrepreneurs produces slightly more. This would raise j's profit enough to allow compensation to all of those entrepreneurs in the chain.

The proposition states that every countably-stable equilibrium is efficient. The proof shows that if a planner controlled production throughout the entire supply chain $\omega\in\Omega_j$ and allocated some quantity of labor l optimally throughout the chain, then it would solve $c(\omega)=\max_{\{l^n(\omega)\}}\prod_{n=0}^{\infty}(\frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}}z^n(\omega)l^n(\omega)^{1-\alpha})^{\alpha^n}$ subject to $\sum l^n(\omega)\le l$. With the optimal allocation of labor across stages, the planner would face the indirect production function $c(\omega)=\prod_{n=0}^{\infty}[z^n(\omega)]^{\alpha^n}l$. Thus, in every equilibrium, each entrepreneur's marginal cost matches the planner's marginal shadow cost of her good.[27]

Proposition 1 also states that, generically, the allocation of goods and labor is essentially unique. The exception is if some entrepreneur j has access to two techniques that would both deliver efficiency q_j. In that case, countable stability does not determine how the entrepreneur splits her production across the two techniques. Given the stochastic process of the arrival of techniques, however, the probability that any entrepreneur has two techniques that deliver exactly the same efficiency is zero. Thus, with probability 1, the set of entrepreneurs with access to two techniques that deliver the same efficiency has measure zero. Finally, payments between buyers and suppliers have no impact on the equilibrium allocation because all entrepreneurs remit their profits to the household.[28]

2.5. *Stability and Payoffs*

This section describes how countable stability restricts the payoffs that entrepreneurs can attain and shows the existence of a class of countably-stable equilibria. Proposition 1 states that countable stability pins down the allocation (the quantities traded), but it is silent about payments from buyers to suppliers, and hence about payoffs. We show now that for each pair in a buyer-supplier relationship, there is a range of payments consistent with stability. In other words, there are many ways buyer-supplier pairs can share their joint surplus.[29] This multiplicity, however, is inessential, because all profits are turned over to the household.

Proposition 2 describes essential upper and lower bounds to the transfers of surplus among entrepreneurs. These bounds apply to a subset of entrepreneurs that we call *acyclic*. A path from j to j' is defined as a finite sequence of distinct entrepreneurs beginning with j and ending with j' along with a corresponding sequence of techniques

$q_j=0,\forall j$. Other examples can be constructed for special cases of Φ. Suppose that Φ is such that entrepreneurs ordered. That is, the entrepreneurs can be divided into sets indexed by $n\in\{0,1,\dots\}$ and any technique for which the buyer is in set n has a supplier in set $n+1$. Suppose that there is a pairwise-stable equilibrium in which entrepreneurs' efficiencies are $\{q_j\}$. Then, for any $t\le 1$, there is a pairwise-stable equilibrium in which entrepreneurs' efficiencies are $\{\tilde{q}_j\}$, where $\tilde{q}_j=t^{\alpha^{-n_j}}q_j$, where j is in set n_j. Such equilibria are robust to deviations by any finite coalition but are not countably stable.

[27]Markups on sales to the household distort the consumption-leisure margin, but since labor is supplied inelastically, there is no impact on the allocation of goods and labor. If labor were supplied elastically or if demand elasticities varied across goods, efficiency would require correcting the monopoly markups on sales to the household.

[28]If entrepreneurs made entry/exit decisions or chose the intensity with which to search for new techniques, the ex post distribution of profit across entrepreneurs would play a more central role.

[29]The range is analogous to a Nash-bargaining set, with one important difference. Here, the split between a buyer and supplier determines how much surplus there is to be split between the supplier and her supplier.

linking each consecutive pair of entrepreneurs. We call entrepreneur j acyclic if, for any other entrepreneur j', there is at most one path linking j to j'. We label the set of acyclic entrepreneurs J^* and define Φ^* to be the set of techniques for which the buyer and supplier are acyclic.[30] Note that with probability 1, the realization of Φ is such that almost all entrepreneurs are acyclic, that is, J^* has measure 1.[31]

To characterize payoffs, it will be useful to express each entrepreneur's profit in terms of surplus, decomposing it into three components: the surplus it generates from sales to the household; transfers of surplus from buyers; and transfers of surplus to suppliers. As shown in Supplemental Material Appendix A.4, in any countably-stable equilibrium, j's profit can be expressed as

$$\Pi_j = \pi_j + \sum_{\phi \in D_j} \tau(\phi) - \sum_{\phi \in U_j} \tau(\phi), \tag{10}$$

where $\pi_j \equiv (p_j - \lambda_j)c_j$ and $\tau(\phi) \equiv T(\phi) - \lambda_{s(\phi)}x(\phi)$. Given the arrangement and individual choices, π_j is the profit from sales of good j to the household when using λ_j as a measure of j's cost and $\tau(\phi)$ is the value of the payment to the supplier in ϕ above the cost of the intermediate inputs when using $\lambda_{s(\phi)}$ as a measure of the supplier's cost.[32]

We define the surplus generated by a technique to be its contribution to all entrepreneurs' profit from sales to the household. The contribution to entrepreneur j's profit from sales to the household is the difference between j's actual profit π_j and that of its best alternative, which we label $\pi_{j\backslash\phi}$. $\pi_{j\backslash\phi}$ is the profit j would earn from sales to the household if it were unable to use any supply chain that passed through the technique ϕ.[33] Let $\mathcal{B}(\phi)$ be the set of all entrepreneurs with supply chains that pass through ϕ. The surplus generated by technique ϕ is then

$$\mathcal{S}(\phi) \equiv \sum_{j \in \mathcal{B}(\phi)} \pi_j - \pi_{j\backslash\phi}.$$

Proposition 2 says that, for almost all techniques—those that are acyclic—countable stability requires that the transfer of surplus from buyer to supplier is bounded by zero

[30] Formally, a *path* from j to j' is a finite sequence of distinct entrepreneurs $\{j^k\}_{k=0}^n$ with $j = j^0$ and $j' = j^n$ and a finite sequence of techniques $\{\phi^k\}_{k=1}^n$, such that, for each $k = 1, \dots, n$, either $j^k = b(\phi^k)$ and $j^{k-1} = s(\phi^k)$ or vice versa (i.e., we ignore the direction of each technique in the sequence). In graph theory, a *connected component* is a subgraph in which any two vertices are connected by a path but no vertex in the component is connected to a vertex outside the component. A *tree* is a connected component in which there is exactly one path linking any two vertices. j is acyclic if, in the graph induced by Φ (and ignoring the direction of each link), j's connected component is a tree. The subgraph comprising entrepreneurs in J^* and techniques in Φ^* is a *forest* (a collection of trees).

One reason stronger statements can be made about acyclic entrepreneurs is that characterizing the "best alternative supply chain" that would be used after a contract is dropped as part of a deviation is easier for entrepreneurs that are acyclic than for those that are not.

[31] For any entrepreneur j, the set of entrepreneurs $\{j'\}$ for which there is a path from j to j' is countable and thus has measure zero. Therefore, the probability that j has two paths to any single entrepreneur is also zero.

[32] Equation (10) follows from cost minimization and pairwise stability, which imply that for every technique, $\frac{wl(\phi)}{1-\alpha} = \frac{\lambda_{s(\phi)}x(\phi)}{\alpha} = \lambda_{b(\phi)}y(\phi)$.

[33] Formally, define $\Omega_{j\backslash\phi}$ be the set of chains to produce good j that do not pass through the technique ϕ. Define $q_{j\backslash\phi} \equiv \sup_{\omega \in \Omega_{j\backslash\phi}} \prod_{n=0}^{\infty}[z^n(\omega)]^{\alpha^n}$ to be the efficiency delivered to j by its best supply chain that does not pass through technique ϕ. Finally, let $\pi_{j\backslash\phi} \equiv \frac{1}{\varepsilon-1}(q_{j\backslash\phi}/Q)^{\varepsilon-1}wL$. Note that if j's best supply chain does not pass through ϕ, then $\pi_j = \pi_{j\backslash\phi}$, that is, ϕ contributes no surplus to j.

and the total surplus generated by the technique. Further, it asserts the existence of a class of countably-stable equilibria that spans those bounds.

PROPOSITION 2: *In any countably-stable equilibrium*, $\tau(\phi) \in [0, \mathcal{S}(\phi)]$ *for each* $\phi \in \Phi^*$. *For any* $\beta \in [0, 1]$, *there exists a countably-stable equilibrium in which* $\tau(\phi) = \beta\mathcal{S}(\phi)$, $\forall\phi \in \Phi^*$ *and* $\tau(\phi) = 0, \forall\phi \notin \Phi^*$.

If there were a technique for which $\tau(\phi) < 0$, the supplier would deviate by dropping the contract, and the entire production chain would shrink production efficiently so that the supplier's marginal cost would remain unchanged. Similarly, if there were a technique for which $\tau(\phi) > \mathcal{S}(\phi)$, the buyer would drop the contract and purchase goods from the supplier associated with its next best technique. That entire supply chain would expand production efficiently so that that supplier's marginal cost is unchanged. Any entrepreneur downstream from the buyer would also adjust production optimally by having her best alternative supply chain efficiently expand production as needed.

The second part of Proposition 2 describes a class of countably-stable equilibria for which there is a natural notion of bargaining power. For each β, there is an equilibrium in which the transfer of surplus from buyer to supplier equals a fraction β of the total surplus generated by that technique.[34]

While bargaining power affects payments among entrepreneurs (and hence transactions data we might observe), it has no impact on the allocation. Proposition 1 implies that the key to solving for aggregate output is to characterize entrepreneurs' efficiencies. Given the network of techniques, Φ, this can be done using (6). One approach is to look for a vector of efficiencies $\{q_j\}_{j\in[0,1]}$ that is a fixed point of (6).[35] However, with a continuum of entrepreneurs, this is neither computationally feasible nor would it be particularly illuminating. Section 3 describes an approach that uses (6) along with the probabilistic structure of the model to sharpen the characterization of the equilibrium allocation.

3. USING THE PROBABILISTIC STRUCTURE TO CHARACTERIZE THE EQUILIBRIUM

The economy's production possibilities—the techniques available to the different entrepreneurs—are stochastic. Given the realization of Φ, the contracting arrangement and individual production choices determine each entrepreneur's efficiency. This section uses the law of large numbers to solve for the *distribution* of efficiencies that is likely to arise given this probabilistic structure. While any individual entrepreneur's efficiency varies across realizations of the economy, the cross-sectional distribution of efficiencies does not. This section shows that the CDF of this cross-sectional distribution is the unique solution to a fixed point problem.

Given Φ, let $F(q)$ be the fraction of entrepreneurs with efficiency no greater than q in equilibrium. This function is endogenous and will need to be solved for. The strategy exploits the fact that F describes the distribution of an entrepreneur's efficiency and that of each of its potential suppliers.

[34]For the properties of the equilibrium described in Section 4 such as the size distribution and the cross-sectional correlations, one can safely ignore sets of measure zero.

[35]Taking logs of both sides gives an operator T, where the jth element of $T(\{\log q_j\}_{j\in[0,1]})$ is $\max_{\phi\in U_j}\{\log z(\phi) + \alpha \log q_{s(\phi)}\}$. T satisfies monotonicity and discounting. If the support of $\log z$ were bounded, the operator would be a contraction on the appropriately bounded function space. This approach to solving the model is useful for simulations with a finite set of entrepreneurs.

What is the probability that an entrepreneur has efficiency no greater than q? This depends on how many techniques she discovers and the efficiency each of those techniques might deliver. The number of upstream techniques available to entrepreneur j, $|U_j|$, follows a Poisson distribution fully described by its mean, M.

Two elements determine the cost-effectiveness of each technique: (i) its productivity, $z(\phi)$, drawn from an exogenous distribution H, and (ii) the efficiency of the supplier, $q_{s(\phi)}$. Since the identity of the supplier is drawn uniformly, the probability that the supplier's efficiency is no greater than q_s is $F(q_s)$.

Let $G(q)$ be the probability that the efficiency delivered by a single random technique is no greater than q. Recalling that the efficiency delivered by the technique is $z(\phi)q^{\alpha}_{s(\phi)}$, this is

$$G(q) = \int_{z_0}^{\infty} F\big((q/z)^{1/\alpha}\big)\,dH(z). \tag{11}$$

To interpret this, note that for each z, $F((q/z)^{1/\alpha})$ is the portion of potential suppliers that, in combination with that z, leaves the entrepreneur with efficiency no greater than q.

Now, the probability that, given all of its techniques, an entrepreneur has efficiency no greater than q is

$$\Pr(q_j \le q) = \sum_{n=0}^{\infty} \underbrace{\frac{M^n e^{-M}}{n!}}_{\Pr(n \text{ techniques})} \underbrace{G(q)^n}_{\Pr(\text{All } n \text{ techniques are} \le q)} = e^{-M[1-G(q)]}.$$

To interpret this last expression, note that if $M[1-G(q)]$ is the mean of a Poisson distribution (the arrival of techniques that deliver efficiency better than q), then $e^{-M[1-G(q)]}$ is the probability of no such techniques.

The law of large numbers implies that, with probability 1, $\Pr(q_j \le q) = F(q)$.[36] Using the expression for $G(q)$ from (11) gives a fixed point problem for the CDF of efficiency F:

$$F(q) = e^{-M\int_{z_0}^{\infty}[1-F((q/z)^{1/\alpha})]\,dH(z)}. \tag{12}$$

This recursive equation is the key to characterizing the equilibrium.

Consider the space $\bar{\mathcal{F}}$ of right-continuous, non-decreasing functions $f : \mathbb{R}^+ \mapsto [0, 1]$, and consider the operator T on this space defined as

$$Tf(q) \equiv e^{-M\int_{z_0}^{\infty}[1-f((q/z)^{1/\alpha})]\,dH(z)}.$$

Supplemental Material Appendix B constructively defines a subset $\mathcal{F} \subset \bar{\mathcal{F}}$ that depends on M and H. Proposition 3 shows that the equilibrium allocation is the unique fixed point of T on $\mathcal{F}$. The proof, contained in Supplemental Material Appendix B, uses Tarski's fixed point theorem to show existence, which also provides a numerical algorithm to solve for it.

[36]The proof of Proposition 3 uses such a law of large numbers for a continuum of random variables described by Uhlig (1996). To use this, one must verify that entrepreneurs' efficiencies are pairwise uncorrelated. That is not immediately obvious in this context, as it is possible that two entrepreneurs' supply chains overlap or that one is in the other's supply chain. However, by assumption, the network is sufficiently sparse that with high probability the supply chains will not overlap: there is a continuum of entrepreneurs, but only a countable set of those are in any of a given entrepreneur's potential supply chains. Therefore, for any two entrepreneurs, the probability that their supply chains overlap is zero.

PROPOSITION 3: *There exists a unique fixed point of T on $\mathcal{F}$, F. With probability* 1, *F is the CDF of the cross-sectional distribution of efficiencies of every countably-stable equilibrium, and aggregate productivity is* $Q = (\int_0^\infty q^{\varepsilon-1}\, dF(q))^{\frac{1}{\varepsilon-1}}$.

The qualitative features of the economy depend on whether the average number of techniques, M, is greater or less than 1. If $M \leq 1$, there are so few techniques that the probability that any individual entrepreneur has access to a supply chain is zero.[37] In the more interesting case in which M exceeds the critical value of 1, there are at least three fixed points of the operator T on $\bar{\mathcal{F}}$, only one of which corresponds to the equilibrium.[38] Because of the multiplicity of fixed points, it is important to restrict the fixed point problem to the function space $\mathcal{F}$, a space for which there is a unique fixed point which corresponds to the equilibrium.

Rather than characterizing how entrepreneurs' efficiencies directly depend on the set of production possibilities Φ, Proposition 3 shows how the cross-sectional distribution of efficiencies depends on the three primitives that generate Φ: the distribution from which technique-specific productivities are drawn, H; the average number of techniques per entrepreneur, M; and α. Section 4 uses this cross-sectional distribution to characterize how these primitives impact productivity and the organization of production.

4. CROSS-SECTIONAL IMPLICATIONS

This section studies how selection of suppliers shapes cross-sectional features of the economy such as how input–output links are distributed across entrepreneurs, the size distribution, and aggregate productivity.

To illustrate these implications, I focus on a parametric assumption that proves to be analytically tractable, allowing for closed-form expressions for the distribution of efficiencies and for aggregate output, and providing a transparent connection between the features of the environment and economic outcomes.

[37] As shown in Supplemental Material Appendix B.4, the probability that an entrepreneur has no supply chains is the smallest root ρ of $\rho = e^{-M(1-\rho)}$. For $M \leq 1$, the smallest root is 1, while for $M > 1$, it is strictly less than 1. This uses a standard result from the theory of branching processes. If $M < 1$, T is a contraction, with the unique solution $f = 1$. A phase transition when the average number of links per node crosses 1 is a typical property of random graphs, a result associated with the Erdős–Renyi theorem. Kelly (1997) gave such a phase transition an economic interpretation.

[38] There are two solutions in which $f(q)$ is constant for all q which stem from the fact that equation (12) is formulated recursively. The first is $f(q) = 1$ for all q, which corresponds to zero efficiency (infinite marginal cost) for all goods; if the marginal cost of every input is infinite, then the marginal cost of each output is infinite as well. This allocation is feasible and pairwise stable but not countably stable. A second, $f(q) = \rho \in (0, 1)$, implies infinite efficiency for entrepreneurs with supply chains and zero efficiency for those without; if inputs have zero cost, then output can be produced at zero cost. As discussed in footnote 25, this does not correspond to a feasible allocation, as it violates chain feasibility.

Each of these correspond to a fixed point of (6). Multiple solutions to first-order conditions are actually a typical feature of models with roundabout production in which a portion of output is used simultaneously as an input. If the same good enters a technological constraint as both an input and an output, the price of that good will be on both sides of a first-order condition. Consequently, the first-order condition will be satisfied if the price takes the value of zero or infinity. One can usually sidestep this issue by finding an alternative way to describe the production technology, that is, solving for final output as a function of primary inputs. Much of the work in the proof of Proposition 3 is in finding and characterizing such an alternative description of production possibilities.

ASSUMPTION 2: *For each producer, the number of upstream techniques with match-specific productivity greater than* z *follows a Poisson distribution with mean* $mz^{-\zeta}$, *with* $\zeta > \varepsilon - 1$.

An economy that satisfies Assumption 2 can be interpreted as the limit of a sequence of economies that satisfies Assumption 1.[39] Consider an economy that satisfies Assumption 1 in which each technique's match-specific productivity is drawn from a Pareto distribution, $H(z) = 1 - (z/z_0)^{-\zeta}$. The shape parameter ζ governs the thickness of the right tail—smaller ζ corresponds to a thicker tail—and there is a minimum cutoff z_0. The sequence of economies takes $z_0 \to 0$, holding fixed the arrival rate of techniques above any productivity level, $M[1 - H(z)]$. Formally, define m to satisfy $M = mz_0^{-\zeta}$. m, a normalized measure of techniques, is defined this way so that for any $z \geq z_0$, the arrival rate of techniques above z is $M[1 - H(z)] = mz^{-\zeta}$. An economy that satisfies Assumption 2 is the limit of a sequence of such economies as $z_0 \to 0$ holding m fixed.[40]

Under Assumption 2, every solution F to equation (12) is the CDF of a Frechet distribution. To see this, note that $H(z) = 1 - (\frac{z}{z_0})^{-\zeta}$ and $M = mz_0^{-\zeta}$ imply that equation (11) becomes

$$M\big[1 - G(q)\big] = mz_0^{-\zeta} \int_{z_0}^{\infty} \big[1 - F\big((q/z)^{1/\alpha}\big)\big] \zeta z_0^{\zeta} z^{-\zeta-1}\, dz$$

$$= q^{-\zeta} m \int_0^{(q/z_0)^{1/\alpha}} \big[1 - F(x)\big] \alpha\zeta x^{\alpha\zeta-1}\, dx,$$

where the second line uses the change of variables $x = (q/z)^{1/\alpha}$. For any q, as $z_0 \to 0$, this expression approaches $q^{-\zeta}$ multiplied by a constant. Label this constant θ, so that equation (12) can be written as

$$F(q) = e^{-\theta q^{-\zeta}},$$

the CDF of a Frechet random variable. The mean of the distribution is increasing in θ, the location parameter, which was defined to satisfy $\theta = m \int_0^{\infty} [1 - F(x)] \alpha\zeta x^{\alpha\zeta-1}\, dx$. Using $F(q) = e^{-\theta q^{-\zeta}}$ and integrating gives $\theta = \Gamma(1 - \alpha) m \theta^{\alpha}$, or more simply,

$$\theta = \big[\Gamma(1 - \alpha) m\big]^{\frac{1}{1-\alpha}}, \tag{13}$$

where $\Gamma(\cdot)$ is the gamma function.[41]

The efficiency distribution inherits the tail behavior of the match-specific productivity draws; in the expression for F, the exponent ζ is the same as that of the Pareto distribution H. As one might expect, θ is increasing in the (normalized) number of techniques per entrepreneur: an entrepreneur with more options to choose from will tend to have higher efficiency. The other determinant of θ is α; its role will be discussed in detail below.

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[39]This type of distributional assumption can be traced back to at least Houthakker (1955), although he did not make a formal argument about a limit. Kortum (1997) showed that his model is well-behaved asymptotically under this type of functional form assumption.

[40]Note that in this limit, the arrival rate of techniques grows without bound ($M \to \infty$) while the distribution of match-specific productivity for any single technique deteriorates ($H(z) \to 1$). The normalization ensures that these occur at the same rate so that the limiting economy is well-behaved. In the limit, the measure of entrepreneurs with no techniques (e^{-M}) shrinks to zero.

[41]$\theta = \Gamma(1 - \alpha) m \theta^{\alpha}$ has three nonnegative roots: the one in equation (13), zero, and infinity. The latter two correspond to the two constant fixed points of equation (12), as discussed in footnote 38.

4.1. *Stars and Superstars*

An entrepreneur's efficiency depends on the arrival of upstream techniques and the availability of efficient supply chains. An entrepreneur's downstream techniques determine the demand for the entrepreneur's good for use as an intermediate input. This section studies how the interaction of upstream and downstream techniques determine an entrepreneur's size and contribution to aggregate productivity.

Proposition 4 characterizes how customers are distributed across suppliers.

PROPOSITION 4: *Suppose Assumption 2 holds*.

1. *Among entrepreneurs with efficiency q, the number of actual customers follows a Poisson distribution with mean $\frac{m}{\theta}q^{\alpha\zeta}$.*

2. *Among all entrepreneurs, the distribution of customers asymptotically follows a power law with exponent $1/\alpha$: $\Pr($# customers $\geq n) \sim \frac{1}{\Gamma(1-\alpha)^{1/\alpha}} n^{-1/\alpha}$.*

PROOF SKETCH: For any technique, let $\tilde{F}(x)$ be the probability that the potential buyer has no *other* techniques that deliver efficiency better than x. Since all such potential buyers have at least one upstream technique, this is $\tilde{F}(x) \equiv \frac{\sum_{n=1}^{\infty} G(x)^{n-1} \frac{e^{-M} M^n}{n!}}{1-e^{-M}} = \frac{F(x)-e^{-M}}{G(x)(1-e^{-M})}$. Note that $\lim_{z_0 \to 0} \tilde{F}(q) = F(q)$.

Consider an entrepreneur with efficiency q. A single downstream technique with productivity z would deliver efficiency zq^α to its potential customer, and $\tilde{F}(zq^\alpha)$ is the probability that the potential customer has no better alternative techniques. Since the number of potential customers is Poisson with mean M, the number of actual customers follows a Poisson distribution with mean $M \int_0^\infty \tilde{F}(zq^\alpha)\, dH(z)$. Under Assumption 2, this equals $m \int_0^\infty F(zq^\alpha) \zeta z^{-\zeta-1}\, dz$, and integrating yields the first result.

Integrating over entrepreneurs of different efficiencies, the mass of entrepreneurs with n customers is $\int_0^\infty \frac{e^{-\frac{m}{\theta}q^{\alpha\zeta}} (\frac{m}{\theta}q^{\alpha\zeta})^n}{n!}\, dF(q) = \int_0^\infty \frac{u^n e^{-u}}{n!} \frac{e^{-[\Gamma(1-\alpha)u]^{-1/\alpha}}}{\Gamma(1-\alpha)^{\frac{1}{\alpha}}\alpha} u^{-\frac{1}{\alpha}-1}\, du$. The second result follows from the fact that a mixture of Poisson distributions inherits the right tail behavior of the mixing distribution as long as that mixing distribution, in this case $\frac{e^{-[\Gamma(1-\alpha)u]^{-1/\alpha}}}{\Gamma(1-\alpha)^{\frac{1}{\alpha}}\alpha} u^{-\frac{1}{\alpha}-1}$, asymptotically follows a power law. *Q.E.D.*

These properties are illustrated in Figure 2. Figure 2(a) shows the average number of customers at each quantile in the efficiency distribution for different values of α. As one would expect, each curve in Figure 2(a) is increasing, indicating high-efficiency entrepreneurs attract more customers.

If α is larger, the high-efficiency entrepreneurs capture an even larger share of customers. There is a single crossing property that is evident in Figure 2(a): with higher α, the expected number of customers is more steeply increasing with efficiency.[42]

Why do higher-efficiency entrepreneurs attract proportionally more customers when α is higher? Recall that the efficiency delivered by a single technique is $z(\phi)q_{s(\phi)}^\alpha$. A technique is more likely to be used by the potential customer when the technique has high match-specific productivity and when the supplier has high efficiency. α is thus the elasticity of the efficiency delivered by a technique to the supplier's efficiency. When α is

[42] Among entrepreneurs at the ϱth quantile of the efficiency distribution (those with efficiency $F^{-1}(\varrho)$), the distribution of customers is Poisson with mean $\frac{\log[1/\varrho]^{-\alpha}}{\Gamma(1-\alpha)}$. Taking the log of this expression, it is easy to see that, across α, this satisfies a single crossing property; the curve is increasing more sharply for higher α.

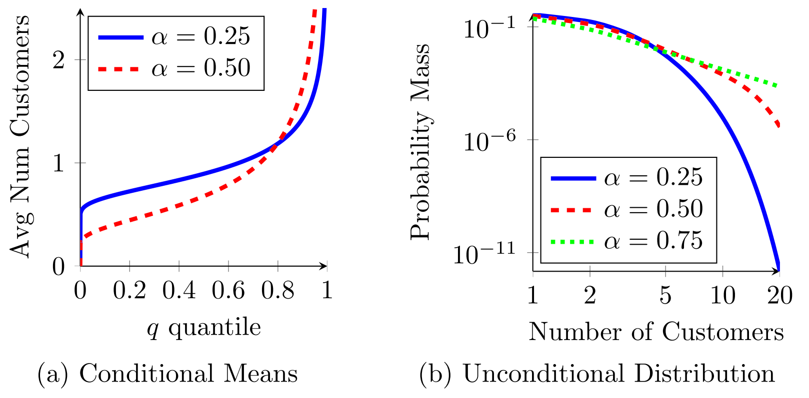


FIGURE 2.—Distribution of customers. Panel (a) shows the mean number of actual customers for each quantile in the efficiency distribution. Panel (b) gives the mass of entrepreneurs with n customers on a log-log plot. Under Assumption 2, the curves in each plot depend only on α.

low, the supplier's efficiency is less important relative to the match-specific productivity because the impact on the technique's efficiency is muted. In contrast, when α is large, higher-efficiency producers are more likely to be selected by each of their potential customers.[43]

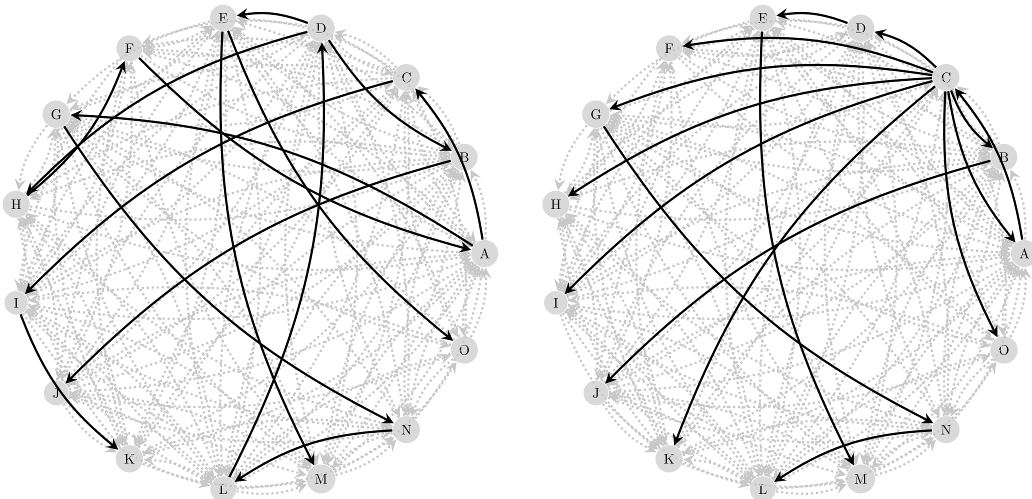
The second result of Proposition 4 describes the unconditional distribution of customers across entrepreneurs, and this is plotted in Figure 2(b) for different values of α. When α is larger, the distribution has a thicker tail, as the high-efficiency entrepreneurs are more likely to attract a disproportionate share of their potential customers. In other words, when α is large, the equilibrium network features more star suppliers—entrepreneurs with many customers. This is illustrated more concretely by Figure 3. In each figure, the set of techniques, Φ, is exactly the same, but α is larger in panel (b).

The model thus provides a mechanism for a skewed cross-sectional distribution of links which complements that of the influential preferential attachment model of Barabasi and Albert (1999). Here, the extra variation stems from selection. Because entrepreneurs differ in marginal cost, they also differ in their likelihood of being selected by potential customers. Some of those entrepreneurs with many potential customers are able to offer lower prices and win over a larger fraction of those potential customers. While the distribution of *potential* customers follows a Poisson distribution, the distribution of *actual* customers exhibits more variation and asymptotically follows a power law.

Both the conditional means and unconditional distribution of customers depend on a single parameter, α. Notably absent is the parameter ζ which governs variation in entrepreneurs' efficiencies: the standard deviation of $\log q$ is $\frac{1}{\zeta}\frac{\pi}{\sqrt{6}}$. In particular, even if there is arbitrarily little variation in entrepreneurs' marginal costs (i.e., ζ is large), the distribution of customers across entrepreneurs would follow a power law with exponent $1/\alpha$, and entrepreneurs with high efficiency would have a disproportionate share of customers.

Why is the distribution of customers independent of ζ? ζ actually plays two offsetting roles. The distribution of customers depends on both the variation in cost of inputs and the likelihood that a potential customer would forgo a technique with better match-specific productivity for one that relies on less expensive inputs. The latter depends on

[43]Another way to see this is to study how the entrepreneurs' efficiencies covary with number of customers. Supplemental Material Appendix C.1 shows that $\frac{\text{Cov}(\log q, \text{\# customers})}{\text{St. Dev.}(\log q)} = \frac{\sqrt{6}}{\pi}\int_0^1 \frac{x^{-\alpha}-1}{1-x}\,dx$, which is increasing in α.



(a) $\alpha = 0.1$ (b) $\alpha = 0.9$

FIGURE 3.—Equilibrium supply chains and α. This figure shows entrepreneurs' choices of techniques. The set of techniques, Φ, is held fixed; the only difference is the value of α. The dark edges represent techniques that are used. $M = 15$ and $H(z) = 1 - z^{-2}$ for $z \geq 1$.

magnitude of variation in technique-specific productivity draws. If all techniques had the same productivity (same z), each potential buyer would simply select the technique with the least expensive input, and the distribution of customers would be heavily skewed toward the most efficient entrepreneurs. In contrast, if all inputs could be purchased at the same cost, each buyer would simply choose the technique with the best fit (highest z). In that case, customers would be relatively evenly distributed across entrepreneurs; the customer distribution would be Poisson. Thus a key driver of the heterogeneity in the distribution of customers is the magnitude of variation in match-specific productivity draws relative to the magnitude of variation in suppliers' marginal costs.

A noteworthy feature of the roundabout nature of production is that the parameter ζ drives variation in both. When ζ is smaller so that there is more variation in productivity draws, there is endogenously more variation in suppliers' cost; the suppliers' costs depend on the match-specific productivity draws of their own upstream techniques.[44] Thus a smaller ζ compresses the distribution of customers by increasing the variation in match-specific productivity draws but also expands the distribution of customers by increasing the variation of potential suppliers' cost of production. These two channels exactly offset.

The Role of Functional Forms

At this point, it is worth commenting on the role of Assumption 2. One reason to focus on Assumption 2 is that it is a special case in which the model has determinate cross-sectional patterns that do not depend on the scale of the economy. For example, the distribution of customers is independent of m. If m were larger, each entrepreneur would have more potential customers. However, on average, the number of customers does not

[44]As mentioned above, the distribution of efficiencies inherits the tail of the distribution of productivity draws.

increase because of a countervailing force: a larger m also raises the number of alternatives available to each of those potential customers. These offset each other, and the scale-free nature of Assumption 2 ensures that they offset exactly.[45]

However, this assumption is not crucial to the main results. Two results can be shown. The first result is fairly intuitive: because customers base their selection of suppliers partly on prices, the distribution of actual customers exhibits more variation than the distribution of potential customers.[46] In other words, selection on prices is a mechanism that leads to increased heterogeneity in engagement in input–output linkages.

The second result is that the prevalence of star suppliers—and the entire distribution of customers—is independent of the variation in entrepreneurs' marginal cost. To see that this result does not depend on Assumption 2, consider the thought experiment of an economy indexed by γ. Relative to the benchmark, in the γ-economy each z is replaced by $\hat{z} \equiv z^\gamma$, so that $H_\gamma(\hat{z}) \equiv H(\hat{z}^{1/\gamma})$; the benchmark economy is recovered by setting $\gamma = 1$. In the γ-economy, the efficiency delivered by supply chain ω is $\prod_{n=0}^{\infty}[\hat{z}^n(\omega)]^{\alpha^n} = \prod_{n=0}^{\infty}[z^n(\omega)^\gamma]^{\alpha^n} = (\prod_{n=0}^{\infty}[z^n(\omega)]^{\alpha^n})^\gamma$. There are two implications. First, $\hat{q}_j = q_j^\gamma$, which means that the dispersion of the log of entrepreneurs' efficiencies is proportional to γ. Second, the ordinal ranking of the supply chains is independent of γ. As a consequence, the economy's equilibrium input–output linkages are independent of γ.

4.2. *The Distribution of Employment*

The distribution of employment across producers has long generated considerable interest from economists.[47] This section builds on Section 4.1 to characterize the determinants of the distribution of employment across entrepreneurs. An entrepreneur produces output to sell to the household for consumption and to other entrepreneurs for intermediate use. Entrepreneur j's choice of employment ultimately depends on the indirect demand for j's good by all other entrepreneurs that produce using supply chains that go through j.

Because an entrepreneur's size depends on the size of each of its customers, the distribution of employment can be characterized recursively. This characterization relies on the fact that a customer's size is sufficient to summarize the indirect demand for j's labor by any other entrepreneurs downstream from that customer. Let $\mathcal{L}(\cdot)$ be the CDF of the overall size distribution and let $\mathcal{L}(\cdot|q)$ be the CDF of the conditional size distribution among entrepreneurs with efficiency q, so that $\mathcal{L}(l) = \int_0^\infty \mathcal{L}(l|q)\,dF(q)$. Proposition 5 describes, for most parameter values, the model's implications for these distributions.

PROPOSITION 5: *Suppose that Assumption 2 holds and that* $\rho \equiv \min\{\frac{1}{\alpha}, \frac{1}{(\varepsilon-1)/\zeta}\}$ *is not an integer. Then right tails of the overall and conditional distributions of employment follow power laws with exponent* ρ:

[45]Recall that Assumption 2 can be interpreted as the limit of a sequence of economies as $z_0 \to 0$. Supplemental Material Appendix C.1 describes how the distribution of customers changes as this sequence converges to its limit.

[46]Formally, the coefficient of dispersion (the ratio of variance to mean) of actual customers is larger than that of potential customers. This follows from the fact that the variance of a Poisson distribution is the same as its mean and the law of total variances. For potential customers, the mean and variance of potential customers is M, so the coefficient is 1. Let n_j be the number of j's actual customers. $\mathbb{E}[n] = 1$, and $\mathrm{Var}(n) = \mathrm{Var}(\mathbb{E}[n|q]) + \mathbb{E}[\mathrm{Var}(n|q)]$. The first term is weakly positive, and the second term is $\mathbb{E}[\mathrm{Var}(n|q)] = \mathbb{E}[\mathbb{E}(n|q)] = 1$.

[47]Among the many, see Lucas Jr. (1978), Jovanovic (1982) Hopenhayn (1992), Axtell (2001), Klette and Kortum (2004), Luttmer (2007), Rossi-Hansberg and Wright (2007), and Arkolakis (2016).

1. $1 - \mathcal{L}(l) \sim K l^{-\rho}$,
2. $1 - \mathcal{L}(l|q) \sim K \frac{m q^{\alpha\zeta}}{\theta} (\alpha l)^{-\rho}$,

where $K \equiv \frac{1}{1-\alpha^{\rho}} \frac{((1-\alpha)\mathbb{I}_{\frac{\varepsilon-1}{\zeta} \geq \alpha} + \alpha \mathbb{I}_{\alpha \geq \frac{\varepsilon-1}{\zeta}})^{\rho}}{\Gamma(1-\rho^{-1})^{\rho}}$.

PROOF SKETCH: Because labor is used to produce output for each destination—for each customer and for the household—rather than working with the overall and conditional size distributions, $\mathcal{L}$ and $\{\mathcal{L}(\cdot|q)\}$, it is easier to work with their respective Laplace–Stieltjes transforms, $\hat{\mathcal{L}}(s) \equiv \int_0^\infty e^{-sl}\, d\mathcal{L}(l)$ and $\hat{\mathcal{L}}(s|q) \equiv \int_0^\infty e^{-sl}\, d\mathcal{L}(l|q)$. The first part of the proof derives fixed point problems for these transforms. Under Assumption 1, the transform of the conditional employment distribution among those with efficiency q is

$$\hat{\mathcal{L}}(s|q) = e^{-s(1-\alpha)(q/Q)^{\varepsilon-1}L} \sum_{n=0}^{\infty} \frac{e^{-M} M^n}{n!} \left[\int_0^\infty \hat{\mathcal{L}}(\alpha s | z q^{\alpha}) \tilde{F}(z q^{\alpha})\, dH(z) \right]^n$$
$$= e^{-s(1-\alpha)(q/Q)^{\varepsilon-1}L} e^{-M \int_0^\infty [1-\hat{\mathcal{L}}(\alpha s|zq^{\alpha})] \tilde{F}(zq^{\alpha})\, dH(z)},$$

where $e^{-s(1-\alpha)(q/Q)^{\varepsilon-1}L}$ is the transform of labor used for the household, $\hat{\mathcal{L}}(\alpha s|zq^{\alpha})$ is the transform of labor used for a potential customer with match-specific productivity z, and $\tilde{F}(zq^{\alpha})$ is the probability that such a potential customer has no better alternatives, as described in the proof of Proposition 4. Imposing Assumption 2 yields $\hat{\mathcal{L}}(s|q) = e^{-s(1-\alpha)(q/Q)^{\varepsilon-1}L} e^{-\frac{m}{\theta} q^{\alpha\zeta}[1-\hat{\mathcal{L}}(\alpha s)]}$. Integrating over entrepreneurs with different efficiencies, and using the change of variables $t = \theta q^{-\zeta}$, gives a single fixed point problem for the transform of the overall distribution of employment:

$$\hat{\mathcal{L}}(s) = \int_0^\infty e^{-s(1-\alpha)\frac{t^{-\frac{\varepsilon-1}{\zeta}}}{\Gamma(1-\frac{\varepsilon-1}{\zeta})}L} e^{-\frac{t^{-\alpha}}{\Gamma(1-\alpha)}[1-\hat{\mathcal{L}}(\alpha s)]} e^{-t}\, dt. \tag{14}$$

The second part of the proof uses a tauberian theorem to relate the slope of the transform at zero to the slope of the right tail of the employment distribution.[48] *Q.E.D.*

Proposition 5 describes the determinants of the employment distribution, and the proof shows that its shape depends only on α and $\frac{\varepsilon-1}{\zeta}$. α matters because it determines the distribution of customers and because it determines how much labor must be used to produce intermediate inputs for a customer. $\frac{\varepsilon-1}{\zeta}$ is a composite of two parameters: ε, the elasticity of substitution across varieties in consumption and ζ, the shape parameter of the distribution of efficiencies. In combination, these parameters determine the shape of the distribution of sales to the household. When ζ is small, there is more dispersion in prices that the household faces, and when ε is high, the household is more willing to substitute toward goods with lower prices.

The first result of Proposition 5 describes the determinants of the right tail of the size distribution. The slope of the right tail is governed by either labor used to make intermediate inputs or by labor used to make final consumption. Across entrepreneurs, the upper tail of labor used to make intermediate inputs for others has a Pareto tail with exponent $1/\alpha$, while labor used to make consumption has a Pareto tail of $\zeta/(\varepsilon-1)$. The proposition

[48]The particular tauberian theorem used is valid when ρ is not an integer. While I have not been able to prove it, a reasonable conjecture is that the requirement that ρ is not an integer can be dispensed with.

says that one of these features will dominate and determine the upper tail of the overall size distribution.

One way to understand this is to note that the conditional transform implies that among entrepreneurs with efficiency q, the average employment is $(1-\alpha)(q/Q)^{\varepsilon-1}L+\alpha\frac{m}{\theta}q^{\alpha\zeta}L$.[49] The first term is labor used to sell goods to the household; the second, to customers (recall $\frac{m}{\theta}q^{\alpha\zeta}$ is the expected number of customers). When α is small, the sales to the household are more important. When α is larger, sales of intermediates becomes more important, but also these sales become more concentrated in star suppliers: $\frac{m}{\theta}q^{\alpha\zeta}$ becomes more steeply increasing in q. When $\alpha > \frac{\varepsilon-1}{\zeta}$, the increases in average employment across entrepreneurs with different efficiencies becomes dominated by the second term, the concentration of sales of intermediates.

The second result of Proposition 5 says that even among entrepreneurs with the same efficiency, the tail of the conditional distribution of employment follows a power law with the same slope as the overall employment distribution. In other words, there is a lot of variation in size even among entrepreneurs with the same marginal cost. Among these entrepreneurs, there is no variation in sales to the household and the distribution of customers follows a Poisson distribution, which has a thin tail. What then drives variation in employment? The granularity of customers: some customers are much larger than others. If, for example, an entrepreneur's customer is a star supplier, the entrepreneur will require a large mass of labor to make intermediate inputs for that customer. Under Assumption 2, the distribution of customer size has a power law tail. The possibility of having a customer at the upper end of the overall size distribution causes the conditional size distribution to inherit this power law tail.[50]

Put differently, suppose $\alpha > \frac{\varepsilon-1}{\zeta}$. The first result of the proposition implies that even when there is little variation in entrepreneurs' efficiencies (ζ large), there can be a lot of variation in size in the cross-section. To use the language of Acemoglu et al. (2012), this is driven by first-order interconnections: some entrepreneurs have many more customers than others. The second result further implies that there is a lot of variation in size even among entrepreneurs with the same efficiency. While there is little variation in first-order interconnections across these entrepreneurs, second-order interconnections drive the variation in size: some of these entrepreneurs' customers have many customers.

Supplemental Material Appendix D presents some preliminary evidence on the relationship between intermediate input shares and the right tails of size distributions. Using data on the size distribution among producers in France and the United States, I ask whether industries with higher intermediate input shares have size distributions with thicker tails. Before proceeding, it is worth sounding a few notes of caution. First, if suppliers have nonzero bargaining power, intermediate input shares are not the same as α, and differences across industries in the dispersion of marginal cost may confound the comparison. Second, the model abstracts from two features of the reality: producers in the real world use more than a single input and there is a lot of variation within industries

[49]The nth moment of a distribution is $(-1)^n$ times the nth derivative of the transform.

[50]Another way to see that the feature that dominates the conditional distribution is the possibility of having a large customer is to compare the scales of the right tails of the overall and conditional employment distributions. Rather than l^ρ, the tail of the conditional size distribution is scaled by $(\alpha l)^\rho$; recall that if a customer uses l units of labor, its supplier uses αl units of labor in producing the intermediate inputs for that customer. In addition, the tail of the conditional size distribution is scaled by $\frac{mq^{\alpha\zeta}}{\theta}$, the expected number of customers. This is closely related to the fact that if N independent random variables X_n are distributed so that $\Pr(X_n > x) \sim x^{-\rho}$, then $\Pr(\sum_{n=1}^N X_n > x) \sim Nx^{-\rho}$.

in intermediate input shares; the simple Cobb–Douglas specification is an enormous simplification and it is not obvious which features would carry over to a more general specification of technology. Third, it is far from obvious what would serve as the best empirical analogue of an entrepreneur in the model; a case can be made for firms, products, or establishments, although each case has its deficiencies. Fourth, a Pareto exponent of $1/\alpha$ would be at odds with the Pareto exponent of the empirical firm size distribution, which is close to 1 (Zipf's law)[51] and perhaps better fits the right tail of the empirical plant size distribution. Finally, treating each industry as a separate economy may be problematic.

With those caveats firmly in mind, I proceed to the cross-industry comparison. First, I use estimates from Di Giovanni, Levchenko, and Ranciere (2011) of tail exponents of the distribution of revenue among firms for each industry in France. In line with the theory, those industries with higher intermediate input shares tend to have thicker tails. Second, I estimate tail exponents for the distribution of employment among establishments and among firms in the United States using data extracts reported by Rossi-Hansberg and Wright (2007). While the point estimates suggest that industries with higher intermediate input shares have employment distributions with thicker tails, the estimates are not precise enough to be distinguished statistically from zero.

4.3. *Matching Patterns*

In environments with matching, one frequently studied property is whether matches tend to exhibit positive or negative assortative matching. Proposition 6 shows that an implication of the model is that whether buyers and suppliers match assortatively depends on which attribute one studies.

PROPOSITION 6: *Under Assumption 2, among techniques that are used in equilibrium*:
1. $\mathrm{Cov}(\log q_{b(\phi)}, \log q_{s(\phi)}) = 0$.
2. $\mathrm{Cov}(n_{b(\phi)}, n_{s(\phi)}) = 0$, *where n_j is the number of j's customers*.
3. $\mathrm{Cov}(l_{b(\phi)}, l_{s(\phi)}) > 0$.

PROOF: We first show that the efficiency of a customer is independent of the efficiency of the supplier. If an entrepreneur with efficiency q_s has a downstream technique with productivity z, the potential buyer selects that technique with probability $\tilde{F}(zq_s^\alpha)$ (defined in Proposition 4). Since the average number of downstream techniques is M, the fraction of actual customers whose efficiency is no greater than q is

$$\Pr(q_b < q | q_s, \text{technique is used}) = \frac{M\displaystyle\int_{z_0}^{q/q_s^\alpha} \tilde{F}\big(zq_s^\alpha\big)\, dH(z)}{M\displaystyle\int_{z_0}^{\infty} \tilde{F}\big(zq_s^\alpha\big)\, dH(z)}.$$

Imposing Assumption 2 and noting that under this assumption $\tilde{F}(q) = F(q) = e^{-\theta q^{-\zeta}}$,

$$\Pr(q_b < q | q_s, \text{technique is used}) = \frac{\displaystyle\int_0^{q/q_s^\alpha} e^{-\theta(zq_s^\alpha)^{-\zeta}} \zeta z^{-\zeta-1}\, dz}{\displaystyle\int_0^{\infty} e^{-\theta(zq_s^\alpha)^{-\zeta}} \zeta z^{-\zeta-1}\, dz} = e^{-\theta q^{-\zeta}},$$

[51] Gabaix (1999) and Luttmer (2007) provided a nice microfoundation for a Pareto exponent close to unity. See also Gabaix (2009) and the references therein.

which is independent of q_s. The second result follows because q_s and q_b are uncorrelated and the probability of being selected as a supplier depends only on efficiency. For the third result, if the buyer uses l_b units of labor, the supplier requires αl_b units of labor to make the intermediate inputs for that supplier. Since the other components of the supplier's labor are uncorrelated with l_b, the covariance of buyers' and suppliers' labor is $\mathrm{Cov}(l_s, l_b) = \mathrm{Cov}(\alpha l_b, l_b) = \alpha \mathrm{Var}(l) > 0$.[52] *Q.E.D.*

Across all techniques, those that use goods produced by higher-efficiency suppliers tend to deliver higher efficiency to the potential buyer. The first result of the proposition shows that, among techniques that are actually used in equilibrium, this correlation is quite different. The key mechanism is selection.

An entrepreneur selecting which of her techniques to use is unlikely to choose one that is associated with a low-efficiency supplier unless the match-specific productivity is unusually high. However, she may choose a technique associated with a high-efficiency supplier even if the match-specific productivity is low; the technique would still be relatively cost-effective because the intermediate inputs can be acquired at a low cost. Together, these imply that, among techniques that are used in equilibrium, those associated with lower-efficiency suppliers tend to have higher match-specific productivity. A consequence is that among techniques that are used in equilibrium, suppliers' and buyers' efficiencies are uncorrelated.[53]

Second, the proposition shows that suppliers with more customers do not tend to have buyers with more customers. This is simply a corollary of the first result: the frequency with which an entrepreneur is selected as a supplier depends only on her efficiency. Since buyers' and suppliers' efficiencies are uncorrelated, there is no correlation in the rate at which they are selected as suppliers.

The third result indicates, however, that the sizes of buyers and suppliers, as measured by employment, are positively correlated in equilibrium. While the supplier's sales to the household and to other customers are uncorrelated with the size of the buyer, the supplier's labor used to produce intermediate inputs for the buyer is perfectly correlated with the buyer's size. If the buyer is large, the supplier will require a large amount of labor to satisfy that buyer's demand for intermediate inputs.

Nevertheless, these results together underscore that care is required when discussing whether matches exhibit positive assortative matching; two natural measures—correlation of buyers' and suppliers' marginal costs or of sizes—yield different answers.

4.4. *The Cost Share of Intermediate Inputs*

In most models with roundabout production, the cost share of intermediates is important because it corresponds to the input–output multiplier. This section shows that, in the environment studied here, the cost share is endogenous, and the next section shows that the cost share may not correspond to the input–output multiplier.

Each entrepreneur makes payments to its supplier and to labor. Entrepreneur j's cost share of intermediates is $\frac{\sum_{\phi \in U_j} T(\phi)}{\sum_{\phi \in U_j} [wl(\phi) + T(\phi)]}$. Similarly, the aggregate cost share of materials is $\frac{\int_0^1 \sum_{\phi \in U_j} T(\phi)\, dj}{\int_0^1 \sum_{\phi \in U_j} [wl(\phi) + T(\phi)]\, dj}$. Recall that a payment to a supplier can be decomposed into variable and fixed components, $T(\phi) = \lambda_{s(\phi)} x(\phi) + \tau(\phi)$, and that pairwise stability implies

[52] Equation (14) implies that the variance of employment is finite only if α and $\frac{\varepsilon - 1}{\zeta}$ are small enough.

[53] The fact that the covariance is exactly zero depends on Assumption 2.

that the cost of labor used with a technique is proportional to the variable component of the payments to the supplier, $\lambda_{s(\phi)}x(\phi) = \frac{\alpha}{1-\alpha}wl(\phi)$. The aggregate cost share therefore equals

$$\frac{\int_0^1 \sum_{\phi \in U_j}\left[\frac{\alpha}{1-\alpha}wl(\phi)+\tau(\phi)\right]dj}{\int_0^1 \sum_{\phi \in U_j}\left[wl(\phi)+\frac{\alpha}{1-\alpha}wl(\phi)+\tau(\phi)\right]dj}.$$

Rearranging and using the labor market clearing condition $\int_0^1 \sum_{\phi \in U_j} l(\phi)\,dj = L$ gives

$$\frac{\alpha wL + (1-\alpha)\int_0^1 \sum_{\phi \in U_j}\tau(\phi)\,dj}{wL + (1-\alpha)\int_0^1 \sum_{\phi \in U_j}\tau(\phi)\,dj}.$$

One immediate result is that the intermediate input cost share is weakly larger than α, because in any countably-stable equilibrium, $\tau(\phi)$ is almost surely weakly positive.[54] Proposition 7 characterizes the aggregate intermediate input cost share under Assumption 2 for the equilibrium in which suppliers receive a fraction β of surplus of each technique.

PROPOSITION 7: *Under Assumption 2, in a countably-stable equilibrium in which $\tau(\phi) = \beta\mathcal{S}(\phi)$, the aggregate cost share of intermediates is*

$$\frac{\alpha+\beta/\zeta}{1+\beta/\zeta}. \tag{15}$$

Average revenue among those with efficiency q is $[\frac{\varepsilon}{\varepsilon-1}(q/Q)^{\varepsilon-1}+\frac{\alpha+\beta/\zeta}{1-\alpha}\frac{m}{\theta}q^{\alpha\zeta}]wL$.

PROOF SKETCH: Since techniques can be enumerated by either their buyers or suppliers, $\int_0^1 \sum_{\phi \in U_j}\tau(\phi)\,dj = \int_0^1 \sum_{\phi \in D_j}\tau(\phi)\,dj = \beta\int_0^1 \sum_{\phi \in D_j}\mathcal{S}(\phi)\,dj$. The rest of the proof shows that $\int_0^1 \sum_{\phi \in D_j}\mathcal{S}(\phi)\,dj = \frac{1}{1-\alpha}\frac{1}{\zeta}wL$.

Define $r_j \equiv \sum_{\phi \in D_j}\mathcal{S}(\phi)$ to be the total surplus of all of j's downstream techniques. Define $v_j \equiv \pi_j^0 + r_j$ to be the surplus of entrepreneur j; if j were unable to produce, v_j would be the total change in profit among j and all entrepreneurs downstream from j. Let $V(q)$ and $R(q)$ be the averages of v_j and r_j, respectively, among entrepreneurs with efficiency q. Among techniques downstream from such entrepreneurs with match-specific productivity z and for which the buyer's best alternative delivers efficiency $\tilde{q}$, the average surplus is $V(\max\{zq^\alpha, \tilde{q}\}) - V(\tilde{q})$. $R(q)$ is found by averaging over realizations of z, $\tilde{q}$, and the number of downstream techniques:

$$R(q) = M\int_{z_0}^{\infty}\int_0^{\infty}\left[V\left(\max\{zq^\alpha,\tilde{q}\}\right)-V(\tilde{q})\right]d\tilde{F}(\tilde{q})\,dH(z),$$

[54] While it is assumed that the labor market is competitive, collective bargaining could drive a wedge between the wage and the marginal cost of labor, pushing down the cost share of intermediate inputs.

where, as in the proof of Proposition 4, $\tilde{F}(\tilde{q})$ is the probability that the buyer has no alternative technique that delivers efficiency better than $\tilde{q}$. Imposing Assumption 2 (which also implies $\tilde{F}(\tilde{q}) \to F(\tilde{q})$) and making the change of variables $u = zq^\alpha$ gives

$$R(q) = m \int_0^\infty \int_0^\infty \big[V\big(\max\{zq^\alpha, \tilde{q}\}\big) - V(\tilde{q})\big]\, dF(\tilde{q}) \zeta z^{-\zeta-1}\, dz = \frac{m}{\theta} q^{\alpha\zeta} \rho,$$

where $\rho \equiv \theta \int_0^\infty \int_0^\infty [V(\max\{u, \tilde{q}\}) - V(\tilde{q})]\, dF(\tilde{q}) \zeta u^{-\zeta-1}\, du$. V can thus be expressed as $V(q) = \frac{1}{\varepsilon-1}(q/Q)^{\varepsilon-1} wL + \frac{m}{\theta} q^{\alpha\zeta}\rho$. Substituting this into the definition of ρ and solving for ρ yields $\rho = \frac{1}{1-\alpha}\frac{1}{\zeta} wL$. Finally, we have that $\int_0^1 \sum_{\phi\in D_j} \mathcal{S}(\phi)\, dj = \int_0^\infty R(q)\, dF(q) = \rho$.

Revenue of entrepreneur j is $\pi_j^0 + \beta r_j + \frac{wl_j}{1-\alpha}$. Average revenue of those with efficiency q is thus $\frac{1}{\varepsilon-1}(q/Q)^{\varepsilon-1} wL + \beta R(q) + \frac{\int_0^\infty l\, d\mathcal{L}(l|q)}{1-\alpha}$. Using the expression for $R(q)$ from above and for $\int_0^\infty l\, d\mathcal{L}(l|q)$ from Section 4.2 gives the result. *Q.E.D.*

If buyers have all of the bargaining power ($\beta = 0$), then the cost share is simply α. In that case, the cost per unit of the intermediate input equals the buyer's (and supplier's) shadow value. If suppliers have more bargaining power, the cost share of intermediate inputs rises. However, the payment to a supplier is limited by the buyer's next best option. Under Assumption 2, $\frac{1}{\zeta}$ is a measure of the variation in match-specific productivities. When ζ is smaller, there is typically a larger distance between the efficiency delivered by each buyer's best and second-best techniques. This raises the surplus to be split between the buyer and the supplier, and increases the payment for intermediate inputs.

4.5. *Aggregate Output*

Aggregate output in the economy is $C = QL$. With the functional forms, aggregate productivity is $Q = (\int_0^\infty q^{\varepsilon-1}\, dF(q))^{\frac{1}{\varepsilon-1}} = \theta^{1/\zeta}\Gamma(1 - \frac{\varepsilon-1}{\zeta})^{\frac{1}{\varepsilon-1}}$. Combining this with the expression for θ from (13) gives an expression for the household's consumption:

$$C = \left[\Gamma\left(1 - \frac{\varepsilon-1}{\zeta}\right)^{\frac{\zeta}{\varepsilon-1}} \Gamma(1-\alpha)^{\frac{1}{1-\alpha}} m^{\frac{1}{1-\alpha}}\right]^{\frac{1}{\zeta}} L. \tag{16}$$

There are several immediate implications. First, as in Kortum (1997), aggregate output increases with more techniques, m.[55] In an economy with more techniques, entrepreneurs tend to have larger sets of supply chains to choose from and, hence, are more likely to have found an efficient one. When ζ is small, the distribution of productivity draws has a thicker upper tail. The exponent $\frac{1}{\zeta}$ scales (exponentially) the productivity of any technique, and consequently each entrepreneur's efficiency. The term $\Gamma(1 - \frac{\varepsilon-1}{\zeta})^{\frac{\zeta}{\varepsilon-1}}$ reflects the household's ability to consume more of less expensive goods. ε, the elasticity of substitution across varieties, measures the household's willingness to substitute toward lower-cost goods, while ζ indicates how much cheaper these lower-cost goods are.

Second, the exponent $\frac{1}{1-\alpha}$ that appears in several places is an input–output multiplier that shows up in any model with roundabout production. α determines the extent to which

[55] In the special case in which $\alpha = 0$, input–output relationships play no role and the expression for aggregate output is the same as Kortum (1997).

lower input prices feed back into lower cost of production. A notable difference between this and the standard input–output multiplier is that, in this context, α is not necessarily the same as the cost share of intermediate inputs; if suppliers have some bargaining power ($\beta > 0$), then α is smaller than the cost share.

A separate, more interesting, role is that α determines the composition of the entrepreneurs supplying intermediate inputs. Recall from Section 4.1 that α determines the frequency with which the lower-cost producers are selected as suppliers. In other words, when α is closer to 1, the lowest-cost producers are more likely to become star suppliers and be more relevant for aggregate production. At a more fundamental level, the supply chains used to produce each good are more likely to be routed through the higher-productivity techniques. Aggregate output is higher because these higher-efficiency techniques are used more intensively. Mathematically, this shows up in the term $\Gamma(1-\alpha)$.[56]

In sum, when α is high, each supplier is able to pass through cost savings to its customers at a higher rate *and* supply chains used in equilibrium are more likely to be routed through higher-efficiency techniques.

4.6. *Identifying Model Parameters*

In this section, I suggest some strategies that might be useful in identifying the parameters of the model. While I do not pursue these strategies here, I discuss what kind of variation might shed light on the structural parameters in this class of models.

As shown by Proposition 7, one may not be able to identify α from cost share of intermediate inputs. Rather, α corresponds to the input–output multiplier. In principle, α could be identified by measuring this multiplier directly. Suppose, for simplicity, that a policy change introduced an iceberg cost of shipping goods from one producer to another or to the household, so that $\delta > 1$ units of a good must be shipped for one unit to arrive.[57] This is isomorphic to dividing all match-specific productivities by a factor of δ, so that the efficiency of every supply chain is divided by a factor of $\delta^{\frac{1}{1-\alpha}}$. α could be identified from studying how aggregate productivity varied with changes in δ: $\frac{d\ln Q}{d\ln\delta} = -\frac{1}{1-\alpha}$.

To identify ζ, one could use the fact that $\zeta + 1$ is the elasticity of substitution across different groups of entrepreneurs in aggregate spending on intermediate inputs. This strategy is borrowed from the trade literature; in Eaton and Kortum (2002), the shape parameter of the Frechet distribution describing producers' productivities is also the trade elasticity.[58] The following example illustrates this idea with a slight departure from the baseline framework. Suppose that entrepreneurs could be randomly divided into K groups $J_1, \dots, J_K$ of sizes $\mu_1, \dots, \mu_K$, respectively. Suppose also that the only alteration to the model is that sales of those in group k are subject to a proportional sales tax at rate $\delta_k - 1$. Buyers would, of course, substitute across suppliers in response to the taxes. As shown in Supplemental Material Appendix C.5, the ratio of total intermediate input purchases from suppliers in group k to intermediate input purchases from suppliers in group

[56]Note that $\Gamma(x)$ is decreasing on $(0, 1)$, with $\lim_{x\searrow 0}\Gamma(x) = \infty$ and $\Gamma(1) = 1$. To get a sense of the magnitude, a change in α from 0.55 to 0.65 increases the standard multiplier $\frac{1}{1-\alpha}$ from 2.2 to 2.9, increases $\Gamma(1-\alpha)$ by a factor of 1.3, and increases $\Gamma(1-\alpha)^{\frac{1}{1-\alpha}}$ by a factor of 3.2.

[57]This is like a proportional sales tax collected in goods where the proceeds are thrown into the ocean.

[58]If one could measure marginal costs directly, one could estimate ζ by studying the dispersion in marginal cost across suppliers. However, in this environment, measuring marginal costs could be quite difficult, as units might not be comparable across producers and marginal costs may differ from average costs because of the transfers of surplus.

k' would be $\frac{\int_{j\in J_k}\sum_{\phi\in D_j}T(\phi)}{\int_{j\in J_{k'}}\sum_{\phi\in D_j}T(\phi)} = \frac{\mu_k\delta_k^{-\zeta}}{\mu_{k'}\delta_{k'}^{-\zeta}}$. Thus changes in tax rates could be used to identify ζ: $\frac{d\log\frac{\mu_k\delta_k^{-\zeta}}{\mu_{k'}\delta_{k'}^{-\zeta}}}{d\log\delta_k/\delta_{k'}} = -\zeta$.[59]

In a similar way, ε could be identified by studying how the household substitutes across goods as relative prices change. Finally, the bargaining power parameter β could be identified using the aggregate cost share of intermediate inputs along with the estimates of α and ζ.

5. CONCLUSION

This paper developed a theory of the formation of an economy's input–output architecture and characterized the implications for the size distribution, organization of production, and productivity. When intermediate goods are more important in production, activity becomes more concentrated in star suppliers, even when differences across producers in marginal cost are arbitrarily small. This raises aggregate productivity as supply chains are more likely to be routed through higher-productivity techniques, and also increases the market concentration in sales of intermediate goods. While research documenting patterns of micro linkages is at an early stage, the model provides both testable implications and an organizing framework to guide empirical work as new data sets emerge.

That being said, the model could be enriched in a number of ways so that it can be mapped more cleanly to data. In the model, techniques use a single input, while most real-world producers use multiple inputs. Similarly, α is assumed to be the same for all techniques, but there is enormous variation in cost shares of materials in microdata. Some entrepreneurs may be more capable than others, so that there is a dimension of productivity that is not tied to particular inputs. Finally, as with many studies with heterogeneous producers, it is far from obvious whether the appropriate empirical analog of an entrepreneur is a firm, an establishment, a product, or something else altogether. Further research could help address these gaps.

A key channel in the model is that the network structure—who buys inputs from whom—matters for aggregate productivity. This channel may be useful in assessing the consequences of the marked changes in sourcing during macroeconomic crises or supply chain disruptions.[60] The mechanism may also be important when studying distortions that may cause producers to use the wrong suppliers, leading them to use lower-productivity techniques or higher-cost inputs.[61] Such distortions may include contracting frictions, state mandates to purchase inputs from particular suppliers, or barriers such as the Berlin Wall. Appropriately modified, the model would provide a natural link between such distortions and aggregate productivity.

[59]The example is relatively simple, and in practice the various groups might represent producers in different industries or in different locations. Each of those wrinkles would suggest modifying the model in the appropriate way to fit the application: entrepreneurs in one location might be more likely to attain techniques that use suppliers in nearby locations than distant ones, or entrepreneurs in one industry, for example, tires, might tend to use suppliers in particular other industries, for example, rubber. Further, if entrepreneurs were in different locations, one might dispense with the assumption of a common labor market and a single representative household. Nevertheless, these wrinkles are easily accommodated, and while the exact expression used to identify ζ might vary with the application, the same general strategy should carry over.

[60]Gopinath and Neiman (2014) and Lu, Mariscal, and Mejıa (2017) documented changes during crises, while Carvalho, Nirei, Saito, and Tahbaz-Salehi (2016) and Barrot and Sauvagnat (2016) studied disruptions following natural disasters.

[61]Jones (2011, 2013) argued that distortions may cause producers to use the wrong input quantities.

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