The Network Origins of Firm Dynamics: Contracting Frictions and Dynamism with Long-Term Relationships *

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Abstract

We study theoretically and empirically how firm-to-firm sales relationships shape firm dynamics and productivity. We first present a parsimonious model of firm dynamics where dynamics arise from the arrival of new potential matches between firms, acting as supply shocks from the perspective of buyers, and as demand shock from the perspective of suppliers. Buyers switch to new suppliers when it is optimal to do so. The model matches the empirical regularities on firm volatility and exit probabilities declining with size, endogenous fat tails in firm growth rates, and some firms with persistently (but not permanently) high growth rates ("gazelles"). We apply the model to a setting in which contracting frictions between firms give rise to long-term relationships. These arrangements improve incentives within the relationship, but firms switch to new suppliers less frequently. This reduces firm dynamism, in the sense that firm sales are less volatile, there is less mean reversion, exit rates are lower, and the right tail of the firm size distribution is thinner. We corroborate these predictions with production data on Indian manufacturing plants and transaction-level data from Pakistan, using variation across regions in court congestion as a proxy for weak formal enforcement and variation across industries in whether the output requires customization. Using a quantitative implementation of our model we show that the dynamic cost of long-term contracts is significant, with the increased court congestion between the state with the fastest courts and the state with the slowest courts reducing aggregate productivity by roughly 15%.

KEYWORDS: Firm Dynamics, Contracting Frictions, Relational Contracting, Misallocation
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1 Introduction

There are systematic differences in firm dynamics across countries at different stages of development. In developing countries small firms, even when productive, remain small while large incumbents tend to stay large, and firms rarely exit the market (Hsieh and Klenow, 2014). One mechanism that may contribute to this phenomenon is that firms choose to source using long-term relational contracts to overcome hold-up problems associated with poor contract enforcement by formal judicial institutions. While these relational contracts have the virtue of improving performance within relationships, they may inhibit firms from switching to cheaper suppliers (Johnson, McMillan and Woodruff, 2002, Hémous and Olsen, 2018). If firms are reluctant to switch suppliers, young, productive firms may not grow as fast as they otherwise would because potential customers tend to stay in their current relationships.

In this paper we study theoretically and empirically how the formation and destruction of firm-to-firm relationships shape firm dynamics and aggregate productivity. We construct a quantitative model in which firm dynamics are driven by a process of matching between potential buyers and suppliers — acting as supply shocks from the perspective of buyers, and as demand shocks from the perspective of suppliers — and which determines the equilibrium distribution of firm sizes and productivities. To validate the model's predictions and to quantify the importance of contracting frictions, we apply the model to the contexts of India and Pakistan, two economies that are characterized by enforcement frictions that, as we show, give rise to long-term relationships and to a low degree of dynamism.

In Section 2 we construct a model of firm dynamics that features relationships between buyers and suppliers, which we argue are key for understanding the patterns of firm dynamics that are present in the data. The model features a continuum of firms that each draw suppliers for their inputs and need to decide when to switch to new suppliers. In general, characterizing this decision of when to switch suppliers can be complicated because of the enormously high dimension of each firm's state. Under a set of assumptions about the matching and productivity process, each firm's cost of production becomes a random walk, which makes the decision tractable because only the current cost of the incumbent and alternative suppliers need to be compared. Demand for firms' output arises endogenously from firms being chosen as suppliers.

A firm's size depends on how many customers it has as well as the size of those customers, which depends on how many customers they have, etc. A firm with a low cost will get selected more frequently by potential buyers that it encounters. Firms grow and shink when they (or their customers) gain or lose customers. A firm exits when it loses its last customer. In Section 3, we show the model is consistent with a number of empirical regularities about firm dynamics beyond the standard ones that motivated canonical models of firms dynamics such as Hopenhayn (1992). Firm volatility declines with size, but more slowly than one would expect if a firm were simply the sum of independent components of similar size. The distribution of firm growth rates has fat right and left tails. Large firms occasionally exit, but the exit rate declines smoothly with size and approaches zero as firm size goes to infinity. The model features gazelles—firms have persistently (but not permanently) high growth rates. The model is parsimonious in the sense that there is just a single shock that drives all of the dynamics. Despite this simplicity, the model is consistent with these empirical regularities because that one shock plays a number of different roles, depending on a firm's relationship to the location of the shock; customers naturally vary in size because some of those customers have accumulated many customers. Changes in size come from acquiring or losing customers, from customers growing or shrinking (because *they* gain and lose customers, etc.). Changes in firm's cost, because it finds a new better supplier (or its supplier finds a new, better supplier), also affect a firm's sales and that of its customers.

Section 4 introduces contracting frictions and relational contracts and describes the model's predictions for firm dynamics. With more severe contracting frictions, firms switch suppliers less frequently. This reduces firm volatility, reduces the rate of mean reversion, reduces the exit rate, and reduces the thickness of the right tail of the firm size distribution.

Section 5 uses production data on Indian manufacturing plants and transactions data on Pakistani firms to test these predictions. We use court congestion as a proxy for weak formal contract enforcement. We show that more congested courts have a larger impact on firm dynamics in industries that produce goods that are relationship-specific (Rauch, 1999). Our approach is to use variation across locations in court congestion and study the differential impact on firms that produce relationship-specific goods vs. firms that produce standardized goods. Thus we control for any local factors that affect all firms, or industry characteristics that would be present in all locations. Our baseline specifications address endogeneity using a variety of fixed affects, and we also employ an instrument for court congestion in the context of India.

Using the transactions data, we find that more severe contraction frictions increase the duration of buyer-supplier relationships more when the the supplier's output is relationship-specific rather than standardized. In line with the model's predictions, we find that more severe contracting frictions reduce firm volatility, reduce the rate of mean reversion, reduce the exit rate, and reduce the thickness of the right tail of the firm size distribution.

Finally, Section 6 studies the impact of slow firm dynamics induced by weak contract enforcement on aggregate output. The model predicts that contracting frictions weaken the relationship between firm cost and firm size, as firms become less likely to switch to good suppliers. We find that in the state with the most congested court, output is 15% lower than it would be if its courts congestion were reduced to the level of the state with the least congested courts.

1.1 Related Literature

Our model is one in which firm dynamics are driven by the formation and destruction of firm-to-firm linkages. Any new link is a new customer of the supplier and a new supplier for the customer. A number of of papers have documented the importance of customer accumulation for firm growth (Luttmer, 2011; Gourio and Rudanko, 2014; Afrouzi, Drenik and Kim, 2020; Argente et al., 2021; Einav et al., 2021). These are consistent with Foster, Haltiwanger and Syverson (2016), who find that firms grow by accumulating demand over time. Here, demand is simply customers that choose to buy a firm's good. Separately, a number of papers have documented using customs data that switching of suppliers is relatively frequent (Gopinath and Neiman, 2014, Lu, Mariscal and Mejía, 2024, Damijan, Konings and Polanec, 2014) and that it is slower for relation-specific inputs (Monarch, 2022). Further, Baqaee et al. (2023) show that switching suppliers is an important channel for changes in a firm's cost. Our paper brings these perspectives together as part of a single unified phenomenon.

Most of the literature on firm-to-firm trade that has focused on the patterns of firm heterogeneity have done so in a static environment: Oberfield (2018), Bernard, Moxnes and Ulltveit-Moe (2018), Eaton, Kortum and Kramarz (2022), Bernard et al. (2022). We introduce dynamics to be able to speak to long-term relationships. Furthermore, our model generates predictions about the impact of contracting frictions on outcomes such as variance of growth rates and exits for which dynamics are essential. The models by Chaney (2014) and Aekka and Khanna (2024) feature firm-to-firm trade, life-cycle dynamics, and a size distribution. In those models, firms have a continuum of customers and suppliers, so size evolves deterministically.

Most models with firm-to-firm trade and dynamics have focused on understanding the pass through of shocks (Lim, 2018, Huneeus, 2018) or recovery from shocks (Miyauchi, 2018) rather than dynamic moments of firm characteristics. Martin, Mejean and Parenti (2023) and Fontaine, Martin and Mejean (2023) incorporate frictions to adjustment of firm-to-firm relationships to study the the impact of trade shocks.

Our paper also speaks to the literature that attempts to explain low aggregate productivity in developing countries. A number of papers study the impact of court congestion on contract enforcement and firm performance, in India (Boehm and Oberfield, 2020, Amirapu, 2021, Chemin, 2012) and elsewhere. The literature that attempts a quantification of those frictions does so from a static perspective. A number of papers discuss firm dynamics in these settings: Hsieh and Klenow (2014) document slow firm dynamics in India. Akcigit, Alp and Peters (2021) posit that this is caused by delegation frictions that make it costly for some firms to grow. In our model, the relational contracts slows the reallocation of customers across firms, similar to how firing costs slow the reallocation of workers across firms in Hopenhayn and Rogerson (1993). The long-term relationships resulting from contracting frictions affect both firm's costs on the supply side, but also act to reduce demand and growth for young productive firms. Finally, our paper builds on the microeconomic literature that emphasizes the role of relational contracts in developing-country contexts (Macchiavello and Morjaria, 2015, Macchiavello and Morjaria, 2021). We study the importance of these types of arrangements in a quantitative model of firm dynamics.

Kwon, Ma and Zimmermann (2023) and Chen (2023) have documented that the right tail of the firm size distribution has gotten thicker in the US over time, and Chen (2023) has shown that the tail is thicker in countries that are more developed. In our model, more severe contracting frictions which result in longer relationships makes the right tail thinner.

2 Simple Model

The economy consists of a representative household, firms that produce (or simply "firms"), and retailers. Firms produce goods using labor and intermediate inputs, and sell their output to other firms and to retailers. In this simple model, retailers produce goods in the same way as firms but only sell goods to households. Retailers compete monopolistically in selling goods to the household. The household supplies labor to firms and purchases goods from retailers.

We first describe an environment in which the economy consists of a single industry of firms. In Section 3.1 we extend the model to allow for many industries.

2.1 Households

There is a representative household with a labor endowment that grows at rate γ, so that the measure of labor available at t is $L_t = L_0 e^{\gamma t}$. The household earns wages and receives dividends from a representative mutual fund that owns the firms.

The household has Dixit-Stiglitz utility across the output of all the retailers. Thus if R_t is the set of retailers at time t and the household consumes y_{jt} from retailer j, the household's period utility is $u(Y_t)$, where the aggregator Y_t is defined as

$$Y_t \equiv \left(\int_{j \in R_t} y_{jt}^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}} .$$

Labor can be used for production or to create new entrants. If a fraction η_t of labor is used to create entrants, a flow $\eta_t L_t \chi$ of new entrants are created. When an entrant is created, there is a fixed probability ζ the entrant becomes a retailer, so that with probability $1-\zeta$ it becomes a producing firm.[1]

2.2 Production

A firm is a variety. To produce, a firm uses a technique. A technique is a triple: a buyer, b, a supplier, s, and a match-specific productivity, z. A technique is a production function for the buyer

$$y_b = A(z_{bs} x_s)^{\alpha} l^{1-\alpha}, \qquad A \equiv \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}$$

where y_b is output of the buyer's good, x_s is units of the supplier's good used as an intermediate input, and l is labor.

At a point in time, a firm has access to a single technique, but is constantly drawing new techniques, comprised of a new potential supplier along with a match-specific productivity associated with using that supplier. When it encounters a new potential supplier, it can either switch to the

[1] The model would remain tractable if we assumed, instead, that entry could be directed toward retailing or producing. We maintain here that entry is undirected so that, in our counterfactuals, we can avoid taking a stance on how the arrival of new potential matches changes when the relative proportion of prospective buyers and prospective suppliers changes.

new one or remain with its old one. Similarly, each new entrant draws a set of techniques that it might use. The set of suppliers is randomly drawn from all existing producing firms, and for each of those suppliers it draws a match specific productivity. As we discuss below, in equilibrium the buyer will select the supplier that delivers the lowest effective cost.

2.3 Static Equilibrium

We begin by characterizing a static equilibrium, and then proceed to characterize a balanced growth path.

We focus on a static equilibrium concept in which there is monopolistic competition among retailers in selling to the household and stable contracting arrangements among firms and retailers.

A contract between a buyer-supplier pair b, s is a quantity of the supplier's good x_{bs} and a transfer payment from the buyer to the supplier, T_{bs}. A contracting arrangement consists of a contract between each buyer-supplier pair. Given a contracting arrangement, a wage, w, and household demand for each retailer's good, each producing firm chooses a quantity of labor, and each retailer chooses a quantity of labor and a price for its variety to maximize profit.

Each firm's static profit is its revenue minus its cost. For a producing firm j that uses supplier s, and has a set of buyers $\mathcal{B}_j$, static profit is

$$\pi_j = \sum_{b \in \mathcal{B}_j} T_{bj} - wl_j - T_{js} \ .$$

For retailer j that sets price p_j and faces household demand $y_j = Y(p_j/P)^{-\varepsilon}$, static profit is

$$\pi_j = p_j y_j - wl_j - T_{js} \ .$$

A contracting arrangement is stable with respect to a coalition of firms if the firms in the coalition do not wish to change the terms of the contracts among them and do not wish to drop contracts with others that are not in the coalition. A contracting arrangement is pairwise stable if it is stable with respect to deviations by a single firm or deviations by a coalition consisting of a buyer-supplier pair. A contracting arrangement is countably stable if it is stable with respect to any countable coalition of firms. A countably stable contracting arrangement is stable with respect to deviations by a coalition consisting of an entire supply chain, but not with respect to deviations by a positive measure of firms (which could undermine the monopolistic competition among retailers).

Proposition 1 summarizes features of any contracting arrangement that is pairwise stable. An object that will be useful in characterizing the equilibrium is each firm's marginal cost. Specifically, given a contracting arrangement, firm j's total cost of producing output y is $\mathcal{C}_j(y) \equiv \min_{l_j} wl_j + T_{js}$ subject to $A(z_{js}x_{js})^{\alpha} l_j^{1-\alpha} \geq y$. We define firm j's marginal cost as $c_j \equiv \mathcal{C}_j'(y_j)$, where y_j is output induced by the contracting arrangement.[2] Another object that will be useful in characterizing

[2]If j is a producer firm then j's output is $y_j = \sum_{b \in \mathcal{B}_j} x_{bj}$ and is directly determined by the contracting arrangement. If j is a retailer, its output is the quantity demanded by the household, $y_j = Y(p_j/P)^{-\varepsilon}$, which depends on the price

equilibria will be the efficiency of the supply chain to produce an input. Suppose a buyer b uses supplier s_1, and that supplier s_1 uses supplier s_2, etc. These buyer-supplier pairs form a supply chain. We define the supply chain's productivity to be

$$q_b = z_{b,s_1} z_{s_1,s_2}^{\alpha} z_{s_2,s_3}^{\alpha^2} \dots$$

Proposition 1 *In any pairwise stable contracting arrangement,*

1. *For any firm j, $wl_j = (1-\alpha)c_j y_j$*
2. *For any buyer-supplier pair, $c_s x_{bs} = \alpha c_b y_b$*
3. *For any buyer-supplier pair, the buyer's marginal cost is*

$$c_b = \left(\frac{c_s}{z_{bs}}\right)^{\alpha} w^{1-\alpha}$$

4. *For any retailer j, $p_j = \frac{\varepsilon}{\varepsilon-1} c_j$*
5. *For any firm j, $\pi_j \geq 0$.*
6. *For any firm j, $c_j \geq w q_j^{-\alpha}$.*
7. *Aggregate output is*

$$Y = (1-\eta)L \int_{j \in R} \left(\frac{c_j}{w}\right)^{1-\varepsilon} dj$$

The first three results show that in any pairwise stable contracting arrangement, supply chains are efficient in that there is no double marginalization. The fourth result shows that the retailers set the usual markup over marginal cost when setting a price for the household. Fifth, no firm cannot earn negative profit; a firm that earned negative profit would deviate by dropping all contracts. Sixth, there is a lower bound on each firm's marginal cost that is determined by the efficiency of the supply chain to produce its input; this is an implication of pairwise stability and feasibility. The seventh result describes aggregate output in the economy. Labor used for production is $(1-\eta)L$, so that aggregate productivity is just the usual Dixit-Stiglitz aggregator of cost across the set of retailers, R. The resulting expression for aggregate output reflects the result that there is no double marginalization in supply chains.

We study countably stable contracting arrangements. These arrangements are stable with respect to deviations by coalitions of countable sets of firms. This means, in particular, that countably stable contracting arrangements are a subset of pairwise stable arrangements. Proposition 2 provides a characterization of countably stable equilibrium.

Proposition 2 *Given a wage and household spending, a contracting arrangement is countably stable if and only if the induced choices satisfy (i) $\pi_j \geq 0$ for all firms j,(ii) $T_{bs} \geq c_s x_{bs}$ for all buyer suppler pairs, and (iii) $c_j = w q_j^{-\alpha}$ for each firm j.*

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j chooses, p_j.

There are many countably stable equilibria. Each equilibrium features the same allocation. However, equilibria differ in how surplus is split across firms. In each retailer's supply chain, the retailer gets profit from charging a markup to the household. This profit is divided across the supply chain according to the transfers in the contracting arrangement. Countable stability guarantees that each participant in the supply chain earns a non-negative share of the surplus of the supply chain, but imposes no more restrictions.

Many notions of equilibrium determine split of surplus using some form of bargaining weights. We assume here that the surplus received by each supplier is fraction α of the surplus received by the buyer.[3] This is equivalent to assuming that each firm's share of the supply chain's surplus is proportional to its share of the supply chain's total expenditure on labor. With these bargaining weights, if a firm spends wl_j on labor, its total revenue is $\frac{\varepsilon}{\varepsilon-1}\frac{1}{1-\alpha}wl_j$ and its payment to its supplier is $\frac{\varepsilon}{\varepsilon-1}\frac{\alpha}{1-\alpha}wl_j$, so its static profit is $\frac{1}{\varepsilon-1}wl_j$.[4,5]

2.4 Dynamics

For each firm, new techniques arrive randomly according to a Poisson process. The identity of the supplier is randomly drawn from all existing firms. The match-specific productivity is random.

We assume that when a firm gets a new potential supplier, there is a spillover from its existing match. In particular, we follow Buera and Oberfield (2020) in assuming that the new match specific productivity is the product of an original component, b, and a component that is inspired by the firm's existing supply chain. So if firm j's initial supply chain has productivity q and a new supplier arrives with a technique with original component b, the match-specific productivity with the new supplier would be $z = qb$.

We assume, that the arrival rate of new techniques with original component greater than b is $\kappa b^{-\beta}$. As a result, the arrival rate of a new supplier that offers a reduction in effective cost larger than x is

$$\phi x^{-\beta}$$

where $\phi \equiv \kappa \frac{1}{|J_t|}\int_{j\in J_t}\left(\frac{c_{jt}}{w_t}\right)^{-\beta} dj$, and $|J_t|$ is the set of producer firms in the cross section at t.

There is no recall. Once a buyer switches to a new supplier, there is no option to switch back to the supplier it left.

[3] Specifically, if the j's total revenue, net of its unit cost, is $\sum_{j\in\mathcal{B}_j}(T_{bj} - c_j x_{bj})$, then j's transfer to its supplier s satisfies $T_{js} - c_s x_{js} = \alpha\sum_{j\in\mathcal{B}_j}(T_{bj} - c_j x_{bj})$.

[4] One attractive property of these bargaining weights is that each firm's expenditure on labor is proportional to its revenue, so that there is a single notion of a firm's "size." We discuss alternative bargaining weights that are consistent with equilibrium in Appendix D.1.4.

[5] Each firm's ratio of revenue to cost is $\frac{\varepsilon}{\varepsilon-(1-\alpha)}$. This is smaller than the usual expression for a markup $\frac{\varepsilon}{\varepsilon-1}$ because the contracts specify both a price and quantity instead of just a price. This allows each buyer-supplier pair to split surplus without distorting quantities, avoiding double marginalization. The buyer's shadow value of the input equals the supplier's marginal cost even though the payment per unit is larger than the suppliers marginal cost.

2.4.1 Discussion of Model Assumption

The mechanics of the model are governed by the random arrival of techniques and one key economic decision: when a new potential supplier arrives, should the firm switch or remain with its current supplier? In principle this is a complicated decision because a firm's state is extraordinarily large.

The main impediment to a simple decision rule is mean reversion in cost. To understand why mean reversion can cause problems, consider the following example. A firm b is choosing between two potential suppliers, its new supplier s_1 and a new potential supplier s_2, that respectively offer the buyer effective costs c_1/z_{bs_1} and c_2/z_{bs_2}. With mean reversion in cost, it might be the case that the supplier who offers a lower effective cost now is likely to offer a higher effective cost in the future. A buyer thinking about which of those suppliers to choose will have to weigh its current demand relative to future demand. If the buyer has a low effective cost but few customers, so that its future demand is expected to be higher than current demand, it may value a low future effective cost more than a low current effective cost. Or if it has a high current cost so that it does not expect to gain many new customers, but many current customers, it may place a high value on current effective cost cost. And the decision gets even more complicated when one considers whether a firm expects its customers' customers to grow, etc.

The critical features that keep the decision simple and tractable are modeling assumptions that ensure that each firm's marginal cost follows a random walk. In that case, a firm should simply choose the supplier that currently offers a lower effective cost, as the distribution of that supplier's effective cost at any point in the future would first-order-stochastically dominate that of the potential supplier that currently offers a higher effective cost. Thus there is never any trade-off between current and future effective cost. When comparing two potential suppliers, current efficiency delivered is a sufficient statistic for stochastic dominance of that supplier in the future.

Lack of mean reversion in supplier cost is in line with the empirical findings of Baqaee et al. (2023). They study how a firm's marginal cost changes when, for reasons unrelated to the firm's own productivity, it stops using one supplier and/or starts using a new supplier. They find that the cumulative change in the firm's marginal cost one year after the switch is very similar to the cumulative change two and three years after the switch, consistent with no mean reversion in supplier cost.

Many of the modeling choices have been made with an eye toward ensuring that firm's cost follow a random walk. The spillovers from current matches to the match productivity of a new match ensure that the arrival rate of new suppliers that deliver a given cost reduction is independent of current cost. And if each supplier's cost follows a random walk, then the distribution of cost reductions further upstream are independent of current costs. The assumption that there is no recall is also important. If a supplier could recall one of its suppliers, than a buyer would want to know whether a supplier had a close alternative to *its* suppliers when choosing whether to switch.

Note also that firms in our model never die, even though they might lose their last customer, in which case they remain dormant and do not produce until they get selected as a supplier again. While dormant they still receive draws of new potential suppliers and buyers. We use the term "exit" to refer to this transition from positive to zero production, i.e., a firm losing its last customer. We will associate this with firm exit in the data. In this way, exit does not introduce any mean reversion in cost.

2.5 A Balanced Growth Path

In this section we construct a balanced growth path in which the share of labor used for entry is constant over time, the mass of firms grows at the rate of population growth, and the distribution of cost among firms in the cross section remains constant. Such an equilibrium will feature a constant interest rate and constant growth of per-capita consumption.

Suppose that a constant fraction η of labor is used to create firms each instant, and that the distribution of cost among firms in the cross section is constant over time.

To simplify the exposition, we normalize the wage to 1. Then let $F(c)$ be the fraction of firms in the cross section with cost weakly less than c. Similarly, let $F_\tau(c)$ be the distribution function of cost among firms of age τ. In particular, $F_0(c)$ is the distribution function among entrants. The flow of new entrants is $\chi\eta L_t$ and L_t grows at rate γ, the fraction of firms at t that are age τ is $\gamma e^{-\gamma\tau}$. Therefore the cross sectional distribution of cost satisfies

$$F(c) = \int_0^\infty \gamma e^{-\gamma\tau} F_\tau(c) d\tau \ .$$

For each firm, the arrival rate of techniques that offer cost reductions is constant over time, $\phi = \kappa \int c^{-\beta} dF(c)$.

2.5.1 Entrants

A new entrant gets many initial draws of techniques and chooses the one that delivers the lowest cost. For each technique, the supplier and the match-specific productivity is random. The number of draws of techniques with match-specific component larger than z follows a Poisson distribution with mean $\kappa_0 z^{-\beta}$. For a technique with match-specific productivity z, the probability that the supplier's cost c_s is low enough that the buyer's cost $\left(\frac{c_s}{z}\right)^\alpha w^{1-\alpha}$ is less than c—that is, the probability that $c_s \leq zc^{1/\alpha}$—is simply $F\left(zc^{1/\alpha}\right)$. Integrating across possible draws of match-specific productivities, the number of techniques that deliver to the buyer a cost less than c is Poisson with mean $\kappa_0 \int_0^\infty F\left(zc^{1/\alpha}\right) \beta z^{-\beta-1} dz$. $1 - F_0(c)$ is the probability of no such draws, or

$$1 - F_0(c) = \exp\left\{-\kappa_0 \int_0^\infty F\left(zc^{1/\alpha}\right) \beta z^{-\beta-1} dz\right\}$$

using the change of variables $u = zc^{1/\alpha}$ and integrating by parts yields

$$1 - F_0(c) = \exp\left\{-\kappa_0 c^{\beta/\alpha} \int_0^\infty u^{-\beta} dF(u)\right\} \tag{1}$$

2.5.2 Changes in Cost

A firm's production cost falls if it finds a new better supplier, if its supplier finds a new better supplier, or if any supplier in the firm's supply chain finds a new better supplier. Since supply chains are infinite, there are an infinite number of events that can reduce the firm's cost.

A firm's cost thus follows a stochastic process called a jump process with infinite activity. In any strictly positive interval of time, almost surely an infinite number of suppliers in the firm's supply chain will find new suppliers, reducing the firm's cost. Nevertheless, most of these events are so far upstream that they have only a small impact on the firm's cost, as the new suppliers account for such a small fraction of the supply chain's value added. That being said, when the firm itself finds a new supplier, or if a supplier not too far upstream finds a new supplier, this can have a big effect on the firm's cost. Altogether, even though there are an infinite number of potential events, the cumulative impact of these events is a well behaved distribution of changes in cost.

We now characterize those changes in cost. For $x \geq 1$, let $M(x,t)$ be the probability that a firm's cost declines by a factor weakly less than x in an interval of length t. Because changes in cost accumulate geometrically, it will be useful to work with the Mellin transforms of these distributions.[6] In particular, define $\varphi^M(s,t)$ to be the Mellin transform of $M(x,t)$, i.e., $\varphi^M(s,t) \equiv \int_1^\infty x^{-s} M_x(x,t)dx$.

Proposition 3 $\varphi^M(s,t) = e^{-\phi t \sum_{k=1}^\infty \frac{s}{\beta\alpha^{-k}+s}}$

The kth term in the summation reflects the possibility of there being a new supplier k steps upstream.[7]

At short horizons, the distribution of changes is lumpy. At longer horizons, the suitably normalized distribution of changes converges to a standard normal, in accordance with the central limit theorem.

Proposition 4 *Let $X_j(t)$ be the proportional decline of firm j's cost over an interval of length t. As t grows large,*

$$\frac{1}{\sqrt{t}}\left(\frac{\log X_j(t) - \frac{\alpha}{1-\alpha}\frac{\phi}{\beta}t}{\frac{\alpha^2}{1-\alpha^2}\frac{2\phi}{\beta^2}t}\right)$$

[6] Similarly to the Laplace and Fourier transform (i.e. the characteristic function), the Mellin transform of a distribution entirely characterizes the distribution: the function $\varphi^F(-s)$ yields all fractional moments s of F. Proposition 3 therefore fully characterizes the distribution of cost changes.

[7] The proof of Proposition 3 uses the (normalized) distribution of changes in cost over short horizons, $m(x) \equiv \lim_{t\to 0}\frac{M(x,t)-1}{t}$. This satisfies $m(x) = -\phi x^{-\beta/\alpha} + m\left(x^{1/\alpha}\right)$; over short horizons, a firms cost declines by a factor larger than x if it finds a new, better supplier that delivers a jump in effective cost larger than $x^{1/\alpha}$ or because its supplier's cost falls by a factor greater than $x^{1/\alpha}$; at short horizons, the possibility of a mixture of the two is negligible. Expanding this gives $m(x) = -\phi\sum_{k=1}^\infty x^{-\beta\alpha^{-k}}$. The Mellin transform of (normalized) changes over short horizons can be found by integrating

$$\phi^m(s) = \int x^{-s}dm(x) = \int s x^{-s-1}m(x)dx = -\phi\sum_{k=1}^\infty \frac{s}{\beta\alpha^{-k}+s}\ .$$

Finally, since changes in cost have i.i.d. increments, the distribution of cost changes over an interval of length t is just the accumulation of the changes in cost over short horizons, $\phi^M(x,t) = e^{t\phi^m(x)}$.

converges in distribution to a standard normal random variable.

2.5.3 The Cross Section

We now use these results to characterize distribution of cost in the cross section. In particular we will characterize the Mellin transforms of the distribution of cost in the cross section and among each cohort,

$$\varphi^F(s) \equiv \int_0^\infty c^{-s} dF(c) \tag{2}$$

$$\varphi^F_\tau(s) \equiv \int_0^\infty c^{-s} dF_\tau(c) \tag{3}$$

We first characterize entrants. (1) can be expressed as $F_0(c) = 1 - \exp\left\{-\kappa_0 \phi^F(\beta) c^{\beta/\alpha}\right\}$. As a result, one can compute the integral in (3) for φ^F_0 directly, giving

$$\varphi^F_0(s) = \kappa_0^{\frac{\alpha}{\beta}s} \varphi^F(\beta)^{\frac{\alpha}{\beta}s} \Gamma\left(1 - \frac{\alpha}{\beta}s\right) \tag{4}$$

where $\Gamma(\cdot)$ is the gamma function.

We next characterize firms of age τ. Firm j's cost is the ratio of the cost at birth to the proportional cost reduction since birth, $c_{jt} = c_{j,t-\tau}/X_j(\tau)$. Since these are independent, the transform of cost among those of age τ is thus

$$\varphi^F_\tau(s) = \mathbb{E}\left[c_{jt}^{-s}\right] = \mathbb{E}\left[c_{j,t-\tau}^{-s}\right] \mathbb{E}\left[X_j(\tau)^s\right] = \varphi^F_0(s) \varphi^M(-s, \tau). \tag{5}$$

Finally, given the age distribution, the Mellin transform of those in the cross section is

$$\varphi^F(s) = \int_0^\infty \gamma e^{-\gamma\tau} \varphi^F_\tau(s) d\tau.$$

Plugging in the expressions for φ^F_τ and φ^M and integrating yields

Proposition 5 *The Mellin transforms of cost in the cross section and among entrants satisfy*

$$\varphi^F(s) = \frac{1}{1 + \frac{\phi}{\gamma} \sum_{k=1}^\infty \frac{s}{s - \beta\alpha^{-k}}} \varphi^F_0(s)$$

and

$$\varphi^F_0(s) = \kappa_0^{\frac{\alpha}{\beta}s} \varphi^F(\beta)^{\frac{\alpha}{\beta}s} \Gamma\left(1 - \frac{\alpha}{\beta}s\right)$$

These transforms are sufficient to characterize the tail behavior of cost in the cross section:

Proposition 6 *The distribution of cost in the cross-section has a power law left tail*

$$\lim_{c \to 0} \frac{\log F(c)}{\log c} = \nu$$

where $\nu > 0$ *is the unique solution of* $\frac{\gamma}{\phi} = \sum_{k=1}^{\infty} \frac{\nu}{\beta\alpha^{-k} - \nu}$.

2.5.4 Aggregate Output

Recall that, in this simple model, the distribution function F describes the distribution of cost both among producing firms and among retailers, as we have assumed that the two types of firms have the same production function and the stochastic process governing the arrival of new potential suppliers is the same. Since labor used for production is $(1-\eta)L_t$, aggregate output is

$$Y_t = \left(|R_t| \int_0^\infty c^{1-\varepsilon} dF(c) \right)^{\frac{1}{\varepsilon-1}} (1-\eta) L_t.$$

Since a fraction ζ of firms become retailers, the mass of retailers at time t is $|R_t| = \int_{-\infty}^{t} \zeta\chi\eta L_{\tilde{t}} d\tilde{t} = \frac{\zeta\chi\eta}{\gamma} L_t = \frac{\zeta\chi\eta}{\gamma} L_0 e^{\gamma t}$. In addition, the integral is simply $\varphi^F(\varepsilon - 1)$. Using these and dividing by population gives output per capita

$$\frac{Y_t}{L_t} = (1-\eta) \left(\frac{\zeta\chi\eta}{\gamma} L_0 e^{\gamma t} \varphi^F(\varepsilon - 1) \right)^{\frac{1}{\varepsilon-1}}.$$

That is, output per capita grows at a constant rate $\gamma/(\varepsilon - 1)$. As in other semi-endogenous growth models, the economy's growth rate is pinned down by population growth, but the level of output per capita along a BGP is determined in the model.

We focus here on the special case in which $\beta = \varepsilon - 1$, so that the elasticity of substitution across producing firms is the same as a the elasticity of substitution among retailers.[8] In this case, output per capita simplifies to

$$\frac{Y_t}{L_t} = (1-\eta) \left(\frac{\zeta\chi\eta}{\gamma} L_0 \right)^{\frac{1}{\beta}} \left[\frac{\kappa_0^\alpha \Gamma(1-\alpha)}{1 + \frac{\phi}{\gamma} \sum_{k=1}^{\infty} \frac{1}{1-\alpha^{-k}}} \right]^{\frac{1}{1-\alpha}\frac{1}{\beta}} e^{\frac{\gamma}{\beta} t}.$$

The numerator of the term in brackets summarizes the level of cost among newborn firms, whereas the denominator captures the declines in cost over the life cycle as well as the age distribution among firms. The exponent $\frac{1}{1-\alpha}$ is the usual input-output multiplier, capturing the fact that a decline in a firms' cost will benefit its customers, its customers' customers, etc. The exponent $\frac{1}{\beta} = \frac{1}{\varepsilon-1}$ captures the impact of the household's willingness to substitute to the lower cost retailers.

3 Firm Dynamics with Firm-to-Firm Trade

In this section we describe several equilibrium outcomes of the model. In particular we show that the model is consistent with a number of well-documented empirical regularities

The model is simple in the sense that there is just one type of shock in the model. Size depends on how many customers a firm has and how big those customers are, which itself depends on how

[8] In the appendix we provide the general expression for when $\beta \neq \varepsilon - 1$.

many customers those customers have, how big those customers' customers are, etc. The number of customers a firm has depends on how many potential customers have arrived, which is random with an arrival rate that is uniform across firms, how many of those potential customers chose to switch to the firm, which depends on the firm's cost (which evolves over time) and the draw of match-specific productivity, which is random, and how long that customer chooses to stay with the firm. Changes in size come from gaining or losing customers or customers growing and shrinking, which comes from them gaining or losing customers, etc. All this is driven by a single shock, the arrival of a new potential match. This is a supply shock to firms downstream from the potential match and a demand shock for firms upstream from the potential match.

Despite the simplicity, the patterns of firm size, growth, and survival that emerge from this model are quite rich. In this section we enrich the model by an industry dimension and calibrate this model to Indian and Pakistani data, and show that it matches a number of well-documented stylized facts about firm dynamics that canonical models of firm dynamics do not speak to.

3.1 Many Industries

We now introduce multiple industries. Each firm in industry $\omega \in \Omega$ produces output using labor and a fixed set of intermediate inputs according to the Cobb-Douglas production function with industry-specific output elasticities of labor and each input, $\alpha_{\omega l}$ and $\{\alpha_{\omega\hat{\omega}}\}_{\hat{\omega}}$ with $\alpha_{\omega l} + \sum_{\hat{\omega}} \alpha_{\omega\hat{\omega}} = 1$. For each industry $\hat{\omega}$ supplying inputs, firm j in industry ω has a single supplier denoted by $\mathfrak{s}(j, \hat{\omega})$ that provides inputs $x_{j,\mathfrak{s}(j,\hat{\omega})}$ and with whom firm j has match-specific productivity $z_{j,\mathfrak{s}(j,\hat{\omega})}$. Firm j's output is

$$y_j = A_\omega l^{\alpha_{\omega l}} \prod_{\hat{\omega}\in\Omega} (z_{j,\mathfrak{s}(j,\hat{\omega})} x_{j,\mathfrak{s}(j,\hat{\omega})})^{\alpha_{\omega\hat{\omega}}}$$

with $A_\omega \equiv \alpha_{\omega l}^{-\alpha_{\omega l}} \prod_{\hat{\omega}} \alpha_{\omega\hat{\omega}}^{-\alpha_{\omega\hat{\omega}}}$.

Retailers now purchase one input from each industry and combines them according to a Cobb-Douglas production function.

At any point in time, each firm has a single supplier in each supplying industry for which $\alpha_{\omega\hat{\omega}} > 0$. For each supplying industry, new potential suppliers arrive randomly and independent of the arrival in other supplying industries. Each new potential supplier comes with a match-specific productivity with an original component and a component inspired by the firm's current supply chain for that input. With multiple industries, the efficiency of the current supply chain for input ω can be expressed iteratively. If firm j uses supplier s for input $\hat{\omega}$, the efficiency of j's supply tree for $\hat{\omega}$ can be expressed as

$$q_{j\hat{\omega}} = z_{js} \prod_{\tilde{\omega}} q_{s\tilde{\omega}}^{\alpha_{\hat{\omega}\tilde{\omega}}}. \tag{6}$$

This formulation remains tractable because within each industry, the log cost of every firm follows the same random walk (although the stochastic process for changes in cost is different for firms in different industries). To see why, note that a weighted average of random walks is still a random walk. If the weights are the same for all firms in an industry, then the log cost of every firm in an industry will evolve according to the same random walk. This happens provided that all firms in each industry produce using the same Cobb-Douglas production function.

Appendix D.1 extends the static equilibrium characterization of Section 2.3 to the multi-industry model. We focus on the equilibrium in which the supplier of input $\hat{\omega}$ to a buyer in industry ω receives transfer of surplus that is a share $\alpha_{\omega\hat{\omega}}$ of the surplus received by the buyer. Again, this has the interpretation that a retailer's profit from sales to the household is split among all firms in its supply chain in proportion to their expenditures on labor.[9] Appendix D.3 provide analogues for the multi-industry model to each of the results from Section 2.5 about the cross-sectional distribution of cost and the distribution of changes in cost along a balanced growth path.

3.2 Calibration

This section describes our calibration, as summarized by Table I.

Parameter	Value	Target	Target value	Data source
Population growth (γ)	0.04	Employment share by age		Hsieh & Klenow (2014)
New technique shape (β)	3.52	Cost reduction from new suppliers	−0.284	Baqaee et al. (2023)
New supplier arrival rate (ϕ)	0.58	Mean relationship length	1.72 years	Pakistan data
Observation threshold	varies	$\frac{\text{Median sales above threshold}}{\text{Threshold}}$	6.36	Pakistan data
Number of retailer firms ratio	60	Annual exit probability	0.05	
Household EoS (ε)	4.52	$\beta+1$		

Table I Parameterization

ϕ is the arrival rate of new suppliers which reduce effective cost. When firms switch suppliers at this rate (i.e., they choose the new supplier whenever it improves cost), the mean relationship length is $1/\phi$. We set $1/\phi$ to match a mean relationship length of 1.72, taken from the Pakistan data (see Section 5).[10]

We turn next to β. Recall that when a firm gets a new supplier, the reduction in effective cost follows a Pareto distribution with shape parameter β. Denote the effective cost of an input as λ. The expected change in log effective cost is $\mathbb{E}[\log(\frac{\lambda}{\lambda'})] = \frac{1}{\beta}$. The overall effect on marginal cost is scaled by the firm's expenditure share of that input. Baqaee et al. (2023) estimate an object using Belgian data that is a close match to this. They estimate that when a firm gets a new supplier that represent $x\%$ of the firm's unit cost (for reasons unrelated to changes in the firm's productivity), the firm's log unit cost declines by -0.284. This corresponds to $\beta = 3.52$.[11]

The rate of population growth, γ, which, along a BGP is equal to the rate of growth of the number of firms, determines the shares of industry employment accounted for by old and young

[9] In this equilibrium, each firm in industry ω has a ratio of revenue to expenditure of $\frac{\varepsilon}{\varepsilon-\alpha_{\omega l}}$. In addition, each firms ratio of revenue to labor of $\frac{w}{\alpha_{\omega l}}\frac{\varepsilon}{\varepsilon-1}$. Thus within each industry, size as measured by revenue is always proportional to size as measured by employment (although the constant of proportionality differs across industries).

[10] Note that κ is the deep parameter that governs the arrival rate of new potential suppliers, while $\phi \equiv \kappa \int c^{-\beta} dF(c)$ is endogenous. For our purposes, however, ϕ is a sufficient statistic which obviates the need to calibrate κ directly. We target the level of ϕ here and will target changes in ϕ in our counterfactuals below.

[11] This parameter plays virtually no role in the patterns of firm dynamics, but affects the magnitude of the losses from misallocation in Section 4.

firms. We set $\gamma = 0.04$ to match these shares in India, as reported by Hsieh and Klenow (2014). See Appendix C.1.1 for details.

We calibrate production function parameters by matching them to observed cost shares from input-output tables which we construct from microdata on intermediate input use in the Indian Annual Survey of Industries (data from 2003-2008). We assume that industries in the model correspond to the 3,509 five-digit manufacturing industries present in these data. We make some minor modifications to the input-output tables that increase the speed with which we can solve the model by making the input-output tables more sparse. See Appendix C.2 for details.

Firms in the retail industry are supplied by intermediate industries. The retailer's expenditure shares are the same as each industry's share of final sales that are observed in the Indian data. Households have Dixit-Stiglitz preferences across retailers. We set the elasticity of substitution across retailers to 4.

The number of retailers relative to those in other industries is important because it determines how granular each retailer is. If there are many retailers relative to producing firms, then most producers would sell to a large number of retailers. This would mean that most producers would be active and would seldom exit. Further, firms would be diversified across customers, limiting the volatility of firm sales. The parameter value ρ_R is the ratio of the number of retail firms to the average number of firms in each non-retail industry. There are $3,509$ non-retail industries, so there are $\rho_R/3509$ retail firms for every non-retail firm in the rest of the economy. We set ρ_R to match an exit rate of 0.05.

Our data consists of firms that are sufficiently large to pass a statutory threshold for reporting. The threshold varies with time but is the same across all industries. We impose a similar threshold so that the moments we calculate are more easily compared with those in the data. In particular, we set a threshold for firms to be "observed" in the model so that the ratio of median output of all firms above the threshold to the threshold itself is 6.36.

Finally, we introduce negative drift so that the log of match-specific productivity declines at rate ϕ/β. This makes it so that the expected change in log cost for each firm is zero. This adjustment to the model allows for a better match to the empirical distribution of employment across cohorts.

We describe the simulation procedure in Appendix C.3.

3.3 Firm Dynamics Facts in the Model and the Data

With the calibrated model in hand, we now study how well it matches some stylized facts about firm dynamics that the literature has documented.

3.3.1 Volatility and Firm Size

The strong version of Gibrat's law (1931)—that the distribution of firm growth rates is independent of firm size—has been repeatedly rejected. In particular, large firms tend to be less volatile than small firms. This fact, pointed out as early as Meyer and Kuh (1957), Hymer and Pashigian (1962), and Mansfield (1962), has been corroborated across many different contexts.[12] Figure 1a shows this size-volatility relationship among firms in the US manufacturing sector from 2010-2015 using FactSet Fundamentals data.[13]

Hymer and Pashigian (1962) suggested that a possible explanation for the negative size-volatility relationship is that firms are composed of subunits and larger firms are composed of more subunits. If each subunit has the same volatility, then larger firms would be less volatile because they are more diversified. This type of mechanism is captured by Klette and Kortum (2004), in which a firm is a collection of products over which the firm has patents, and each product evolves independently.[14]

In the model here, a firm's size depends on its sales to its customers. Conditional on cost, whether a firm gains or loses one customer is independent of the customer's size, and whether a one customer gains or loses customers is largely independent across customers. As emphasized by Kramarz, Martin and Mejean (2020), since larger firms tend to have more customers they are also more diversified.

A second important feature is that the size variance relationship is less sharp then what would would be predicted by a model where a firm were composed of i.i.d. subunits. As noted by Hymer and Pashigian (1962), in that case, volatility would decline at rate of $\sigma(S) \propto S^{-1/2}$. But the rate of decline is typically slower. Stanley et al. (1996) indeed find a log linear relationship, $\sigma(S) \propto S^{-\alpha}$, among firms in Compustat with $\alpha \approx 0.15$, while Yeh (2023) finds an estimate close to 0.25 using the US Longitudinal Business Database. The slopes of the fitted lines in Figure 1a are in this range: 0.18 for employment, .25 for sales.

We are aware of two types of explanations. First, as suggested by Hymer and Pashigian (1962) and adapted by Stanley et al. (1996), there is some correlation of shocks across units.[15] A second explanation, advanced by Amaral et al. (1998), Sutton (2002), and Herskovic et al. (2020), is that individual segments vary in size, with some segments much larger than others. As Gabaix (2011) later explained, when individual components are granular, so that the size distribution of

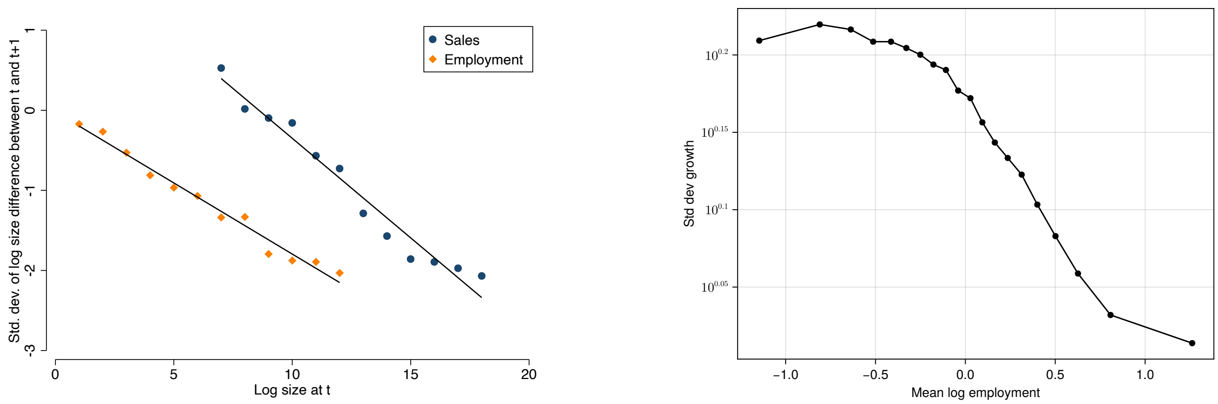
[12]See Yeh (2023) and Coad (2007) for good reviews of the literature.

[13]Firms in this dataset are either publicly traded or have issued a security that are traded, such as corporate bonds.

[14]A number of papers, such as Lentz and Mortensen (2008), Akcigit and Kerr (2018), and Garcia-Macia, Hsieh and Klenow (2019), build on Klette and Kortum (2004) and feature firm dynamics driven by product innovation (and sometimes other types of innovation) at their core. One challenge for this class of explanations are findings from Koren and Tenreyro (2013) and Yeh (2023). They respectively find that the size-volatility relationship is basically unchanged when comparing firms with the same number of segments (Compustat data) or 6-digit products (US Census data), or among firms with a single segment/product. It is of course possible that the relevant subunits are not well captured by segments/products. Large firms also have more establishments than small firms, which may lead to some diversification across establishment-level shocks, but Yeh (2023) finds that the size-volatility relationship holds even after controlling for the number of establishments.

A second explanation is that volatility depends directly on age as it would in a Jovanovic (1982), and the size-volatility relationship simply reflects the correlation of age and size. While volatility does indeed decline with age, Yeh (2023) shows that the size-variance is nearly unchanged when controlling for age, either parametrically or using cohort fixed effects. Yeh (2023) advances a third explanation that the pass through of productivity shocks into price declines with firm size. In this view, i.i.d productivity growth rate shocks translate into less than one-for-one changes in sales/employment because of imperfect pass through of cost into price. There is a demand system with the right match between pass-through and size that delivers the size-variance relationship.

[15]Sutton (2002) examines segments data of Compustat firms and argues that the correlation of growth rates across segments is too weak to explain the flatness of the relationship.



(a) Data

(b) Model

Figure 1 Firm Size and Volatility

Note: Panel (a) shows the relationship between the standard deviation of firm growth rates and firm size among firms in the US manufacturing sector from 2010-2015 from FactSet Fundamentals. Panel (b) shows the relationship between firm size (demeaned by industry) and volatility in the calibrated model.

components has a fat tail, the law of large numbers kicks in more slowly, so that the standard deviation of growth rates declines more slowly than $S^{-1/2}$. As an extreme example, if almost all of a firm's sales are from a single subunit, then its volatility will be almost the same as a small firm with a single subunit, no matter how big the large firm is.

In the model, volatility declines with log size at a rate smaller than 1/2 mostly for this second reason. Customers vary in size. Some larger firms have few customers, and a firm with many customers may have most of its sales accounted for by few customers. While large customers are likely to be less volatile, gaining or losing a large customer is a large event. Those shocks to large customers are not diversified with shocks to smaller customers. In line with Kramarz, Martin and Mejean (2020), a natural prediction of this model is that conditional on size and number of customers, firms with a higher Herfindahl index of sales to customers will be more volatile.[16]

Table II shows the size-volatility relationship in the Pakistani data (first five columns) and contrasts it with the same predictions in simulated data. The first column shows the usual negative relationship between volatility and size. The second column shows that volatility declines with number of buyers, although with a bit less explanatory power. The third column shows that accounting for number of buyers weakens the size volatility relationship by about a third. The fourth column shows that, conditional on size, firms whose sales are concentrated on few customers—as measured by the HHI of sales across customers—are more volatile. Finally, the fifth column shows that even controlling for size and the distribution of sales across buyers, a firm is less volatile when its buyers are larger (and hence less volatile), as measured by a HHI where sales are weighted by

[16] In line with the first rationale for the weaker size variance relationship than 1/2, there is some correlation in growth rate across a firm's customers that comes from a firm's own changes in cost. In our calibration, this is less important quantitatively.

buyer size.[17]

In the simulation, the sales-volatility relationship is somewhat stronger than in the data.[18] That said, the match of relative magnitudes between the data and the calibrated model are remarkably similar, as the coefficients from the regressions using simulated data are roughly double those using actual data. Volatility declines faster with number of customers than with sales, but number of buyers has less explanatory power; once one controls for the number of buyers, the volatility-sales relationship weakens by roughly a third; the HHI better accounts for volatility than number of buyers; and conditional on the distribution of sales across buyers (as captured by the HHI), having larger buyers reduces sales.

	Data					Simulation				
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
$\overline{\log(\text{Sales})}$	-0.138		-0.092	-0.105	-0.103	-0.3021		-0.2424	-0.2259	-0.2256
	(0.0018)		(0.0025)	(0.0022)	(0.0022)	(0.0007)		(0.0009)	(0.0008)	(0.0008)
$\overline{\log(\text{Buyers})}$		-0.217	-0.111				-0.4962	-0.1845		
		(0.0031)	(0.0042)				(0.0014)	(0.0018)		
$\overline{\log(\text{HHI})}$				0.152	0.202				0.3179	0.4224
				(0.0055)	(0.0067)				(0.0017)	(0.0112)
$\overline{\log(\text{HHI (weighted)})}$					-0.051					-0.1058
					(0.0037)					(0.0112)
Fixed Effects										
Industry	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Statistics										
R^2	0.263	0.244	0.286	0.287	0.289	0.7667	0.7393	0.7713	0.781	0.781
R^2-within	0.197	0.175	0.221	0.223	0.225	0.2674	0.1814	0.282	0.3123	0.3124
Observations	23,034	23,034	23,034	23,034	22,552	538,784	538,784	538,784	538,784	538,784

Standard errors in parentheses. The dependent variable is the log standard deviation of $\log \text{sales}_{t+1} - \log \text{sales}_t$.

Table II Determinants of firm growth volatility

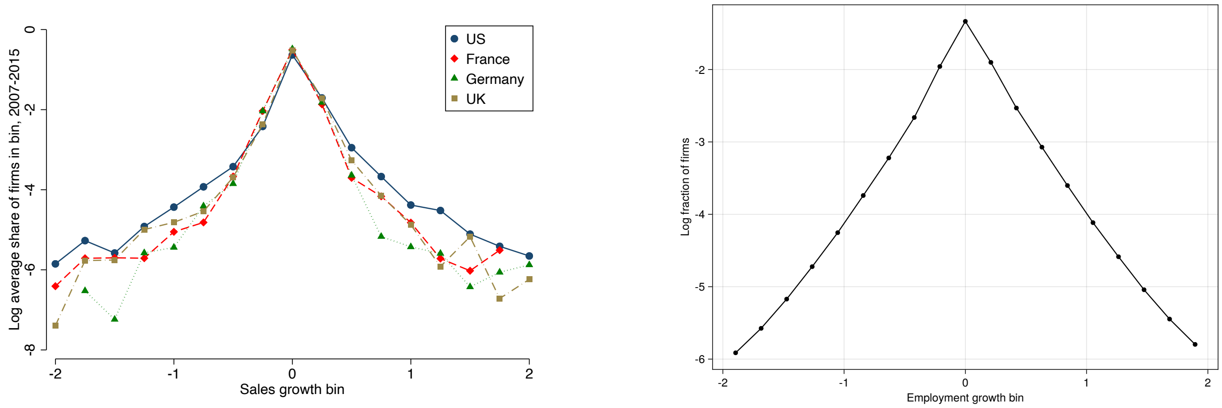
Note: The table shows regressions of the log standard deviation of $\log \text{Sales}_{t+1} - \log \text{Sales}_t$ (calculated within firms, over time) on the respective firm characteristic, averaged across time t. Buyers is the number of firms that purchase a positive amount from the firm in the given year. The HHI for firm j is defined as $\sum_b (\text{share of sales}_{jb})^2$. The HHI(weighted) for firm j is defined as $\sum_b (\text{share of sales}_{jb})^2 (\text{Size}_b)^{0.2}$, where the exponent 0.2 is motivated by the baseline size-volatility relationship. Columns (1) to (5) use data from Pakistan, 2011-2018; columns (6) to (10) use data simulated from the model.

3.3.2 Fat Tails of the Growth Rate Distribution

Another striking feature of firm growth rates is that the distribution features fat tails. This is true for the unconditional distribution, as can be seen in panel (a) of Figure 2 as well as conditional on initial size.

[17] The HHI for firm j is defined as $\sum_b (\text{share of sales}_{jb})^2$. The HHI(weighted) for firm j is defined as $\sum_b (\text{share of sales}_{jb})^2 (\text{Size}_b)^{0.2}$, where the exponent 0.2 is motivated by the baseline size-volatility relationship.

[18] There could be two reasons the size-volatility relationship in the model is stronger in the model than in the data. First, it could be that the model is missing an important element that mediates the size-volatility relationship, such as the possibility of multi-product firms; the model is, of course, quite stylized along other margins. A second possible reason is that there is a mismatch between size as measured in the model and size as measured in the data due to, e.g, the discreteness of shipments or partial year effects, which would dampen the relationship in the same manner as classical measurement error.



(a) Data

(b) Model

Figure 2 The distribution of firm growth rates

Note: Panel (a) shows the distribution of firm growth rates among manufacturing firms in various countries using data from FactSet Fundamentals, 2010-2015. Panel (b) shows the distribution of growth rates in the calibrated model.

Some studies have argued that the distribution of changes in log size follows a Laplace distribution (back to back exponentials), e.g., Stanley et al. (1996), Bottazzi and Secchi (2003, 2006), and Bottazzi, Kang and Tamagni (2023). This would mean that on a log-log plot such as in Figure 2, the distribution is piecewise linear. Some studies find even fatter tails than Laplace.

The model here has the property that the distribution of growth rates features fat tails. Extreme increases or decreases in size can come from gaining or losing a very large customer. Thus even conditional on size, there can be extreme growth events.

3.3.3 Firm Exit

Another well-known fact about firm dynamics is that larger firms are more likely to survive than smaller firms, Mansfield (1962), Hall (1987), Evans (1987*b*,*a*). Some models, such as Hopenhayn (1992) or Luttmer (2007), feature a simple cutoff rule, so that a firm exits when its size drops below the cutoff. This pattern of exit predicted by these models is, of course, much more stark than in the data. In the data, smaller firms are, in fact, more likely to exit. But there are many very small firms that survive, and some very large firms that exit.

A number of papers intend to capture this in a simple way by incorporating both exogenous exit with a fixed opportunity cost of operations, and an exogenous death shock that is i.i.d. across firms and over time, which causes a firm to exit. Again, this does not capture the gradual decline of exit rates with size.

In this model, a firm exits once its last customer switches to an alternative supplier. Some large firms may exit when they have only one large customer and that customers switches away to a new supplier. But most large firms will have many customers, so the probability of exit is small. The probability of exit is smoothly declining in firm size because the probabilities of having a single

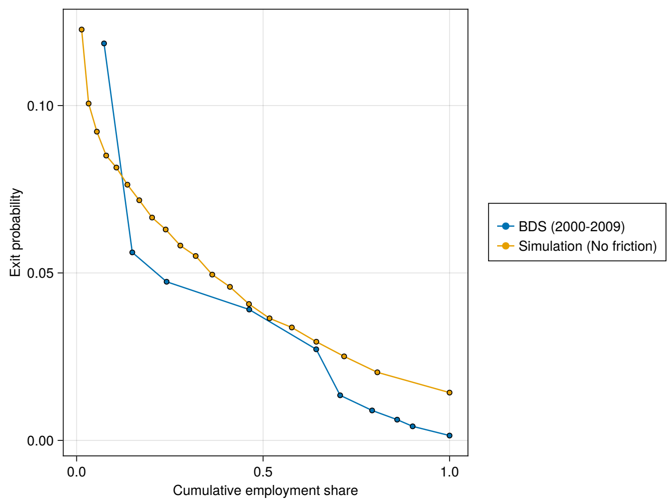


Figure 3 Firm Size and Probability of Exit

Note: This figure shows exit rates for US firms using BDS data and exit rates as predicted by the calibrated model.

customer and of not gaining any additional customers are both smoothly declining in firm size.

This logic is also used by Garcia-Macia, Hsieh and Klenow (2019) who use a model in the spirit of Klette and Kortum (2004) but with dispersion in market size across products. They observe only survival and changes in employment, but they discipline the dispersion in market size across products using how the exit rate varies with firm size.

In Appendix B.2, we show that number of buyers is a good predictor of exit rates among firms in Pakistan. In a horse race, number of buyers has very similar predictive power for exit as size. (R-squared of 9% and 10%, respectively). Further, once we include the number of buyers, the additional predictive power from size is small, rising from 9% to 11%.[19]

3.3.4 Gazelles

Many models of the firm size distribution or of the wealth distribution have trouble matching the quickness with which individuals end up in the right tail (Luttmer, 2011, Gabaix et al., 2016).[20] Luttmer (2011) proposed introducing a persistent (but not permanent) state that he labeled quality

[19]Note that the model predicts that number of buyers is not a sufficient statistic for a firm's exit rate, and that the exit rate should be declining in size even when conditioning on the number of buyers. For example, consider two firms, each with one customer. Suppose that the first firm's customer is large because *it* has many customers while the second firm is small because it has only a single customer. The first firm will exit if its customer switches away. The second firm will exit if its customer switches away or if its customer's customer switches away. Thus even among these two firm with the same number of customers, the larger one is less likely to exit. In Appendix B.2 we show the same regression specifications using data generated by the model. The number of buyers is a better predictor of exit than sales, but, conditional on the number of buyers, sales adds some predictive power.

[20]According to Luttmer (2011), the median age of firms above 10,000 employees is 75 years. He argues that a model in which the volatility of firm growth is independent of size can get close to this when the volatility is calibrated to that of small firms, but models that match the volatility of large firms predict that the largest firms would be much older than in the data.

that determined the firm's growth rate. Gabaix et al. (2016) call this heterogeneity in expected growth rates "type dependence." This made it possible for some young firms to move into the right tail of the firm size distribution quickly. And random transitions from high quality to low quality made it possible for the right tail of firm size distribution to follow a power law. In a similar vein, Sterk, Sedláček and Pugsley (2021) estimate a statistical model of firm size and age. They emphasize the role of a firm's unobserved "ex ante" type, which they argue determines most of the dispersion in firm size among a cohort in its first year and almost half of the dispersion at age twenty. Firms that exhibit persistently high growth rates are often called "gazelles".

The model here naturally exhibits these features. Firms differ in growth trajectories partly because they differ in production cost; a firm with a low production cost expects to acquire customers more quickly. A firm with an especially low production cost may experience persistently high growth rates. Further, as in Luttmer (2011), this "high-growth" state is likely to be persistent but not permanent, as a firm eventually reaches a size where expected outflows of customers matches expected inflows.[21]

While there is no consensus on the definition of a gazelle or on a procedure to detect the presence of gazelles, we propose here a crude such procedure. Firms expand by adding customers and contract when customers leave. Given a firm's unit cost, there is an expected size at which the expected inflow of customers equals the expected outflow, and customers are roughly average size. A firm that is close to its expected size is likely to fluctuate around its expected size, yielding negative autocorrelation in growth rates. A firm that is far from its expected size is likely to gradually converge to its expected size, yielding positive autocorrelation in growth rates. We first ask whether positive growth in one year predicts positive growth in the next year. The set of firms that have positive growth in the first year are likely to be a mixture of the two groups, so positive growth should not be a strong predictor of subsequent growth. We next ask how well positive growth in n consecutive years predicts positive growth in the subsequent year. The higher is n, the more the composition of firms with n consecutive years of positive growth is tilted toward those that are gradually converging to their expected cost and away from those that are fluctuating around their expected cost. As a result, we expect that having n consecutive years of positive growth to more strongly predict positive subsequent growth when n is higher. Figure 4 shows estimates from separate regressions of positive growth on an indicator for n-consecutive years of positive growth and industry fixed effects, with n varying from one year to twelve years in our simulation, and one year to five years in our data on Pakistan (we are limited by the length of the panel). We can do this exercise for growth in sales or for growth in number of customers. Both the data and the simulation suggest the presence of gazelles, in that the four curves are upward sloping: n consecutive years of positive growth covaries more strongly with positive growth in the subsequent year when n is larger.

Newborn firms in particular are likely to be far from their expected size because older firms

[21] Any model with strong enough adjustment frictions and otherwise standard firm dynamics will feature gazelles. Customer accumulation is one such friction. Bilal et al. (2022) argue that hiring frictions work as well.

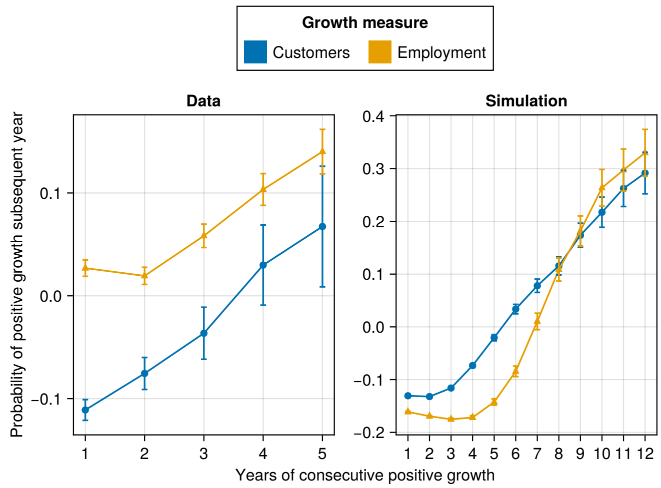


Figure 4 Detecting Gazelles

Note: Each point shows the estimates from a regression of an indicator of positive growth on an indicator of positive growth in each of the last n years and industry fixed effects. The left panel shows results from Pakistan VAT data, with n varying from 1 to 5, while the right panel shows the results from the simulation of the model, with n ranging from 1 to 12.

with a low cost have had time to accumulate customers and converge to their expected size. Since newborn firms are likely to be converging gradually to their expected size whereas older firms are likely to be fluctuating around their expected size, we would expect the autocorrelation of growth rates to decline with age. In line with this, Coad, Daunfeldt and Halvarsson (2018) find that autocorrelation of growth rates is positive for young firms and negative for older firms, providing evidence that some "young firms experience a sudden burst of growth shortly after entry, and that soon afterwards their growth rates slow down and become more erratic."

To get at this prediction, Table III shows regressions of an indicator of positive growth in one year on an indicator of positive growth in the previous year, interacted with age. As the second row of the table shows, younger firms with positive growth are more likely to experience positive growth in the subsequent year than older firms with positive growth.[22]

[22] Coad, Daunfeldt and Halvarsson (2018) caution that the fat tails of firm growth rate distribution complicate the measurement of autocorrelation of growth rates. Our use of an indicator function of positive growth is one approach that is robust to fat tails of the growth rate distribution. Coad, Daunfeldt and Halvarsson (2018) use median regressions. In Appendix B.3 we use the same quantile regression as specified in Coad, Daunfeldt and Halvarsson (2018) on model-generated data. The result gives the same message as Table III: autocorrelation declines with age.

Table III Autocorrelation and Age

	Dependent variable: Positive sales growth between t and $t+1$			
	Data		Simulation	
	(1)	(2)	(3)	(4)
Positive sales growth between $t-1$ and t	0.0972**	0.0926**	-0.120**	-0.052**
	(0.015)	(0.015)	(0.003)	(0.003)
Pos. sales growth between $t-1$ and t × Log Age	-0.0227**	-0.0250**	-0.014**	-0.025**
	(0.0046)	(0.0045)	(0.001)	(0.001)
Log Age	0.00366	0.00563	-0.021**	-0.008**
	(0.0040)	(0.0040)	(0.001)	(0.001)
Log Sales		0.0171**		-0.052**
		(0.00090)		(0.000)
Industry FE	Yes	Yes	Yes	Yes
R^2	0.0149	0.0203	0.032	0.065
Within R^2	0.00182	0.00734	0.028	0.061
Observations	65717	65717	651,480	651,480

Standard errors in parentheses. Columns (1) and (2): data from Pakistan; only one cross-section (firms with positive sales in 2017) used.
$^{+}$ $p < 0.10$, * $p < 0.05$, ** $p < 0.01$

4 Contracting Frictions and Long Term Relationships

In this section we introduce the contracting frictions and relational contracts. When courts are less efficient and thereby inhibit the enforcement of contracts, firms will prefer to use relational agreements with their suppliers and will switch suppliers of relationship-specific goods less frequently. In particular, for users of relationship-specific inputs, the arrival rate of new suppliers of those inputs will be lower when enforcement is poor. As we will show in Section 5, this view will be consistent with data from India and Pakistan.

At this point, we introduce the decline in the arrival rate of new suppliers when contracting frictions are worse as a behavioral assumption. In Subsection 4.1 we discuss a potential microfoundation for this behavior as resulting for the use of relational contracting as a substitute for courts. A relational contract is a repeated game, and there are many equilibria. At this point, we can show for a special case of the model that a uniform decline in κ for relationship-specific inputs is indeed an equilibrium.

4.1 Microfoundation

A contract between a buyer-supplier pair specifies, in addition to a quantity and a transfer, a level of defectiveness $\delta \in [0, 1]$. With commitment, surplus between the buyer and supplier is maximized at $\delta = 0$. The supplier can produce a defective input. Doing so reduces the supplier's cost, but increases the probability that the output will be defective.

If the output is defective, the buyer has a claim on the supplier. This claim can be enforced in court. But delay in court reduces value of payment, and the cost of the delay is proportional to value of transaction.

In a one shot game, both the buyer and supplier would anticipate that the supplier would make a defective input. Even though the price paid by the buyer would reflect this, the defectiveness still reduces the static surplus from the relationship.

As usual, there is an equilibrium in which the static Nash outcome is played at each instant. But there is another, equilibrium that pareto dominates. In this relational contract, the supplier chooses $\delta = 0$, and the buyer chooses a lower arrival rate of new suppliers (which is observable to supplier, but not the court). Doing this makes the relationship likely to last longer, raising the supplier's surplus from the relationship in a way that is backloaded.

The relational contract is enforced with trigger strategies. If the supplier does not customize, the buyer does not reduce arrival of new suppliers. If the buyer does not reduce arrival rate, the supplier stops customizing. The supplier's punishment for defective inputs is that the relationship ends faster *and* enforcement in court. As a result, better formal enforcement reduces the need for the buyer to make the relationship last longer.

4.2 Slow Firm Dynamics

To explore the implications of weak contract enforcement, we turn to a numerical simulation of the model. There is only one change in the model, a reduction in the arrival rate of new suppliers of relationship-specific inputs. But this manifests itself in a number of ways.[23]

For each figure below, we simulate an economy under two scenarios, one with court congestion and one without. In each, only a subset of industries produce relationship-specific inputs. In the scenario with more court congestion, any firm that uses relationship-specific inputs sees a slower arrival rate of new suppliers of those inputs. In all figures below, we plot statistics for the subset of industries that produce relationship-specific inputs, to compare firm dynamics for those industries across the two economies. This will make the predictions comparable to our empirical difference-in-difference strategy that we will use below.

In the first scenario, labeled "no friction," court congestion is such that the average case age is one year, corresponding roughly to the fastest courts in India. In the second, labeled "with friction," courts are more congested and the average case age is four years, corresponding roughly to the slowest courts in India. Our empirical findings below (Table IV) indicate that for each additional year of court delay, relationships among buyer-supplier pairs where the supplier produces relationship-specific inputs last about 0.25 years longer compared to relationships where the supplier produces standardized inputs.

Volatility—First, we show how contracting frictions change the overall distribution of firm growth rates. Figure 5 shows the density of changes in log size among all firms in industries that produce relationship-specific goods. With more severe contracting frictions, the distribution

[23] For now, we hold entry fixed in counterfactuals. While the adjustment of entry will affect aggregate output, it will not affect patterns of firm dynamics that we document in this section. We are currently working on incorporating changes in entry.

of firm growth rates is more concentrated. There are more firms that have smaller growth rates, and fewer firms with extremely large or extremely small growth rates. This is a natural consequence of slower gain and loss of customers. Figure 6 shows the impact of relational contracting on firm-level

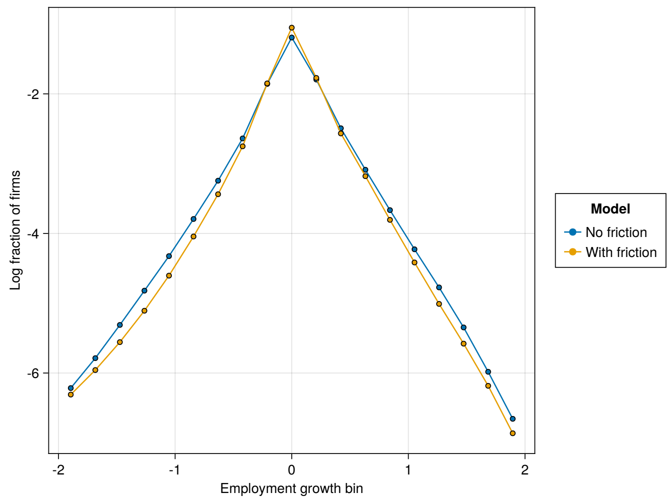
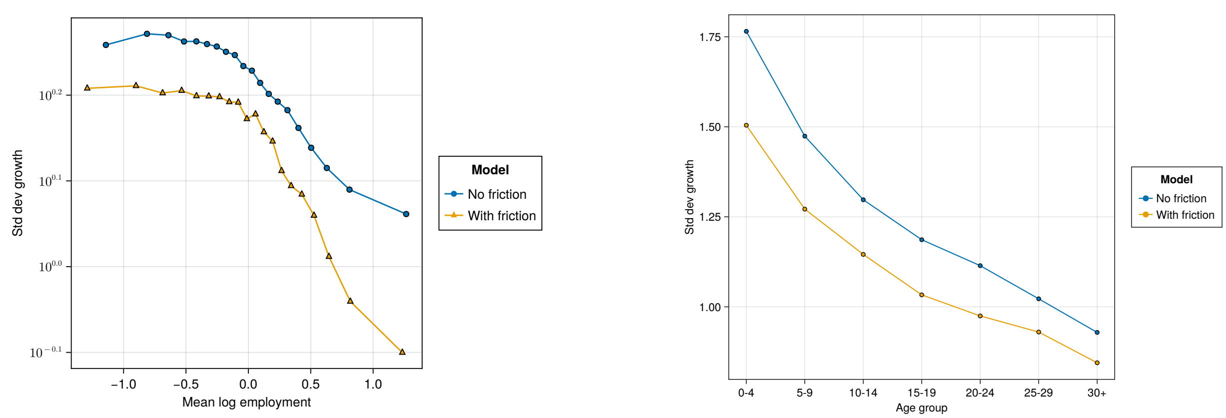


Figure 5 Distribution of Changes in Log Size

Note: This figure shows the density of changes in log size among firms in industries that produce relationship-specific industries.

volatility. With more severe contracting frictions, the standard deviation of changes in log sales is lower, as firms gain and lose customers more slowly. Panel (a) shows lower volatility conditional on size, and panel (b) shows lower volatility conditional on age.



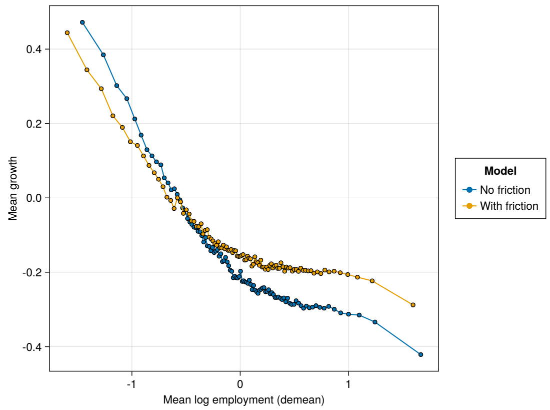
(a) Volatility by Size

(b) Volatility by Age

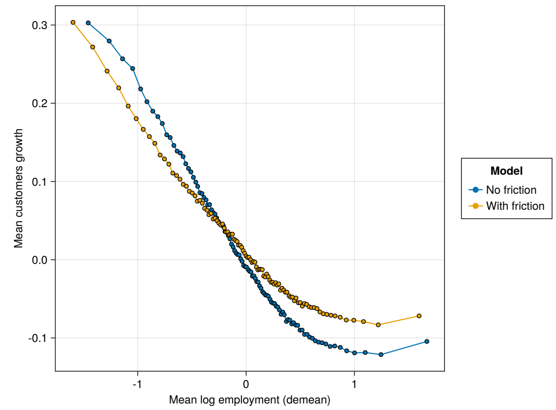
Figure 6 Volatility

Note: For each size or age bin, this figure shows the standard deviation of the log change in size among firms in industries that produce relationship-specific industries.

Mean Reversion—Figure 7 shows that relational contracting leads to slower mean reversion. The two panels show the change in log size conditional on initial size, where change in log size is measured either by sales in panel (a) or by number of customers in panel (b). In either case, small firms grow more slowly, as customers are less likely to switch to them. And large firms shrink more slowly, as they are more likely to keep each customer for longer.



(a) Sales



(b) Number of Customers

Figure 7 Mean Reversion

Note: This figure shows changes in log sales and changes in the log of number of customers for firms of various initial sizes.

Size Distribution—Figure 8 shows the right tail of the within-industry size distribution for industries that produce relationship-specific inputs on a log-log plot. Here, a steeper curve means a thicker right tail. The figure shows that relational contracting leads to a firm size distribution with a thinner right tail. Without the contracting friction, the firms that are in the right tail of the firm size distribution are ones that have a low cost and have been lucky enough to have had many potential customers arrive. With long-term relationships, potential customers do not arrive as quickly, so that firms with a very low cost cannot attract new customers as quickly, and therefore do not grow as large.

Exit—Figure 9 shows exit rates among those that produce relationship-specific goods, by size and by age. With imperfect enforcement, exit rates are lower. A firm exits when it loses its last customer. With relational contracting, firms are less likely to lose their customers in equilibrium, so that a firm that is down to its last customer is less likely to lose that customer and hence less likely to exit.

To summarize, in industries that produce relationship specific inputs, when enforcement is worse, firms have (i) a lower variance of growth rates; (ii) less mean reversion in size; (iii) a less skewed size distribution; (iv) a lower probability of exit.

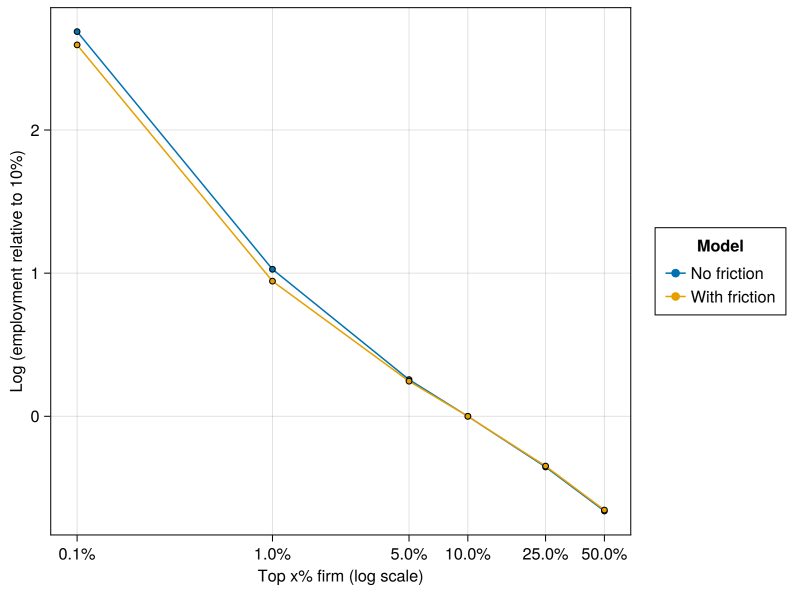
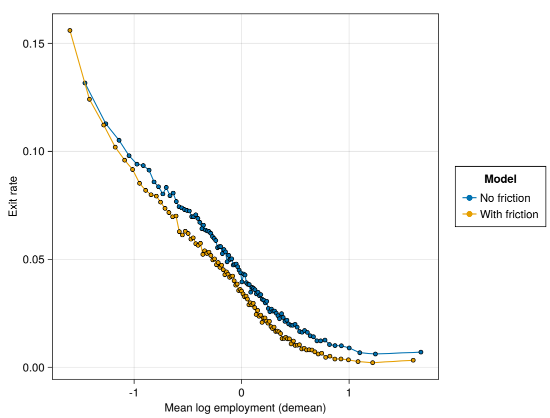
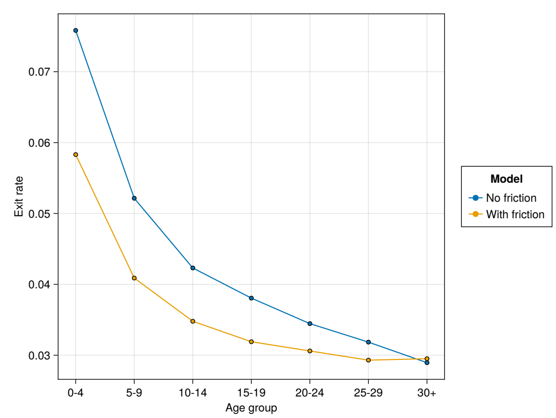


Figure 8 Right Tail of the Firm Size Distribution

Note: This figure shows changes in log sales and changes in the log of number of customers for firms of various initial sizes.



(a) Exit Rates By Size



(b) Exit Rates by Age

Figure 9 Exit Rates by Size and Age

Note: This figure shows exit rates for industries that produce relationship-specific inputs with perfect contract enforcement and with imperfect enforcement. Panel (a) shows exit rates by size. Panel (b) shows exit rates by age.

5 Testing the Model's Predictions

In this section we use data from India and Pakistan to assess the predictions of the model. We first describe the data and the setting, discuss our approach, and then proceed to contrast the model's predictions with empirical patterns in the data.

5.1 Data and Approach

Production and Transactions Data—We use two complementary datasets. First, we use data from India's Annual Survey of Industries (ASI), an unbalanced annual panel of manufacturing plants from India's formal manufacturing sector. In any year the survey covers all plants with more than 100 employees, and about a fifth of all plants that have between 20 and 100 employees (10 and 100 employees if the plant uses power). We use ASI rounds from 1990 to 2015.

Second, we use monthly firm-to-firm transactions data from VAT records in Pakistan. The data encompasses all transaction between formal firms that are subject to and registered for VAT: importers and wholesalers, as well as manufacturers and retailers above 5-10m Pakistani rupees in revenue in the previous year.[24] The data contains monthly transactions for the fiscal years 2011-2012 to 2017-2018, which we aggregate to the annual (fiscal year) level.

Court Congestion—We use data on the average age of civil cases that are pending, at the end of the calendar year 2016, in Indian High Courts, as constructed by Boehm and Oberfield (2020). These measures vary at the state level, from less than one year in Goa and Sikkim, to about four and a half years in Uttar Pradesh and West Bengal. While data on lower-level courts is available and would offer potentially more variation to exploit, Indian firms typically have the (sometimes *de jure*, often *de facto*) ability to bypass lower courts, making High Courts the most relevant ones for enforcement of contract cases.

For Pakistan our measure of court congestion is, similarly to India, the average age of pending civil cases, which we construct from various reports of the *Judicial Statistics of Pakistan* (National Judicial Policy Making Committee, 2021). In contrast to India however we construct this for district courts due to the lower number of provinces (the most closely corresponding administrative unit to Indian states in Pakistan). We also have no indication that parties are able to bypass district courts in Pakistan. Most of our data on Pakistani courts pertains to the year 2020, but we use within-province variation for some states from earlier years (see Appendix A for details).

Approach—The regressions below study the differential impact of court congestion on industries that produce relationship specific goods relative to industries that produce standardized goods. For relationship-specific goods the firms potentially face a hold-up problem, which an effective court system can resolve. All regressions include both industry and region (states in India, districts in Pakistan) fixed effects. This is to control for the impact of factors such as the level of development,

[24] The exact threshold for VAT registration varies over the years, and some sectors or ownership structures are excluded. Balboni, Boehm and Waseem (2024) provide summary statistics of the data and compare it to statistics from national accounts.

population density, the state of technology, to the extent that they have a similar impact on standardized and relationship specific industries.

One might naturally worry that court congestion is endogenous, or that it might be correlated with local features such as the level of development that may differentially impact firm dynamics for firms that produce relationship-specific goods relative to those that produce standardized goods. For India, we use two additional approaches, following Boehm and Oberfield (2020). First, we employ the log age of the court as an instrumental variable for court congestion, based on the fact that congestion increases over time, courts start out uncongested, and the creation of new courts is linked to political subdivisions of states, which is unrelated to congestion or industry structure. Second, we control for the interaction of relationship-specificity with a variety of local features that may be correlated with court congestion. We discuss the latter in Section 5.7.

Each of the two data sources offers advantages and disadvantages. For the ASI data from India we have precise information on the 5-digit products sold by the plant, whereas in the Pakistan data we have only the two-digit industry of the firm, and we cannot see the products in each transaction. There is also more variation in court congestion across states within India, and we have an instrument for court congestion. The advantage of the VAT data from Pakistan is that we can see the firm-to-firm transactions, and the data comes at the firm level rather than the plant level. We show regressions using the data that is more suitable for the purpose (i.e. generally India, unless we need information on transactions, buyers, suppliers, or exit rates, in which case we use data from Pakistan), and show results with the other dataset (whenever possible) in the Appendix.

5.2 Court Congestion and Relationship Length

We first provide evidence on how court congestion affects the length of relationships. The dependent variable in Table IV is the time elapsed between the date of the first transaction and the date of the last transaction of the firm pair, in years. We use the data from Pakistan, where the average duration of relationships is 1.72 years.

When a firm-to-firm transaction is between firms in different districts, the firms may have the freedom to sign contracts that will be enforced in either the buyer's or the supplier's district. In column (1) of Table IV, we assess the impact of court congestion in the buyer's district, the supplier's district, and the minimum of the two. In each case, we find that a more congested court is associated with longer duration when the supplier produces a relationship-specific good compared to when the supplier produces a standardized good.

The first three columns use a measure of relationship specificity from Rauch (1999). The last two columns used a measure of enforcement intensity that is specific to the supplier industry-buyer industry pair from Boehm (2022), based on the frequency of litigation between buyers and suppliers in those industries.

	Dependent variable: Age of Relationship (in Years)				
	(1)	(2)	(3)	(4)	(5)
Age of pending cases (S) × RelSpec_S	0.225**				
	(0.045)				
Age of pending cases (B) × RelSpec_S	0.0638				
	(0.045)				
Age of pending cases (Min(B,S)) × RelSpec_S		0.281**	0.264**		
		(0.032)	(0.041)		
Age of pending cases (Min(B,S)) × $\text{EnforcementIntensity}_{b,s}$				0.0228*	0.0258*
				(0.011)	(0.013)
B × S Industry FE	Yes	Yes	Yes	Yes	Yes
B District FE	Yes	Yes		Yes	
S District FE	Yes	Yes		Yes	
S District × S Industry FE			Yes		Yes
B District × B Industry FE			Yes		Yes
R^2	0.0630	0.0636	0.0929	0.0625	0.0922
Observations	2140189	2142616	2141943	2142616	2141943

Standard errors in parentheses, clustered at the origin-destination district level.
$^+$ $p < 0.10$, * $p < 0.05$, ** $p < 0.01$

Table IV Relationship Duration and Court × Enforcement Intensity interaction

RelSpec_S is a industry-level average of Rauch (1999)'s conservative measure of relationship-specificity (where a good is considered relationship-specific if it's not traded on an organized exchange, and there is no reference price). $\text{EnforcementIntensity}_{b,s}$ is $z^{(1)}$ from Boehm (2022), scaled to have a standard deviation of one.

5.3 Firm Volatility

We next turn to manufacturing data from India's ASI to assess the predictions described in Section IV. Table V shows the impact of court congestion on firm volatility. The dependent variable is the standard deviation of annualized, residualized sales growth within each state-industry-year. These sales growth rates are residualized with respect to age, year, state and industry.

The first two columns estimate coefficients using ordinary least squares, while the third and fourth columns show IV regressions using the age of the court as an instrument for court congestion. The second and fourth columns control for average growth of those firms. In all cases, more severe contracting frictions reduce the volatility of firm size.

5.4 Mean Reversion

Table VI shows the impact of contracting frictions on the rate of mean reversion. The first row shows the baseline rate of mean reversion for plants in industries that produce standardized goods. The coefficient is negative, meaning that larger firms grow slowly. The second row shows the differential degree of mean reversion for firms that produce relationship-specific goods and are located in states with worse enforcement. Across specifications, the point estimate is positive, meaning that the rate of mean reversion is smaller when contracting frictions are present.

	Dependent variable: $\sigma(\Delta \log \text{Sales})_{d\omega}$			
	(1)	(2)	(3)	(4)
Avg age of civil cases × Rel. spec.	-0.0177*	-0.0187*	-0.0401*	-0.0385*
	(0.0089)	(0.0088)	(0.016)	(0.016)
$\overline{(\Delta \log \text{Sales})_{d\omega}}$		-0.273**		-0.273**
		(0.024)		(0.024)
State FE	Yes	Yes	Yes	Yes
5-digit Industry FE	Yes	Yes	Yes	Yes
Estimator	OLS	OLS	IV	IV
R^2	0.287	0.302	-0.000369	0.0207
Observations	7574	7574	7574	7574

Regression at the state × industry level. Only state-industry cells with more than 5 observations used.

Table V Lower variance of sales growth when frictions are large

	Dependent variable: Change in log Sales					
	(1)	(2)	(3)	(4)	(5)	(6)
$\log \text{Sales}_{t-1}$	-0.403**	-0.427**	-0.555**	-0.403**	-0.436**	-0.583**
	(0.011)	(0.025)	(0.037)	(0.012)	(0.028)	(0.038)
$\log \text{Sales}_{t-1} \times$ Age civ. cases × relspec	0.00709+	0.0206*	0.0249+	0.00687	0.0256*	0.0405*
	(0.0037)	(0.0096)	(0.015)	(0.0044)	(0.012)	(0.019)
Plant × 5-digit Industry FE	Yes	Yes	Yes	Yes	Yes	Yes
State FE	Yes			Yes		
Year × Previous Year FE	Yes			Yes		
Age FE		Yes	Yes		Yes	Yes
Industry × District × Year FE		Yes			Yes	
Industry × District $\times(t, t-1)$ FE			Yes			Yes
Method	OLS	OLS	OLS	IV	IV	IV
R^2	0.457	0.636	0.671	0.256	0.250	0.278
Observations	204518	78053	51401	204518	78053	51401

Standard errors in parentheses, clustered at the state × industry level.

Table VI Mean Reversion

5.5 Size Distribution

Table VII shows the impact of contracting frictions on the skewness of the plant size distribution. To measure skewness in the right tail of the plant size distribution, we use a statistic that measures the slope of the log-rank, log-size plot, following Chen (2023). In particular, for any two quantiles of the overall size distribution, S_0 and S_1, we compute skewness of the size distribution in a state-industry-year as

$$\frac{\log(\text{Share of plants above } S_1) - \log(\text{Share of plants above } S_0)}{\log S_1 - \log S_0}$$

The various columns of the table use different combinations of quantiles of the overall plant size distribution (25th, 50th, and 75th, 90th).

Across specifications, we find that more severe contracting frictions are associated with less skewed plant-size distributions.

	Dependent variable: Skewness of log Sales					
	(1)	(2)	(3)	(4)	(5)	(6)
Relspec x Court Congestion	-0.360*	-0.671*	-0.799**	-0.624+	-1.312*	-0.905
	(0.168)	(0.287)	(0.294)	(0.349)	(0.598)	(0.578)
R^2	0.540	0.435	0.554	0.001	0.000	0.007
State FE	Yes	Yes	Yes	Yes	Yes	Yes
5-digit Industry FE	Yes	Yes	Yes	Yes	Yes	Yes
Estimator	OLS	OLS	OLS	IV	IV	IV
Statistic	25-75	50-75	50-90	25-75	50-75	50-90
Observations	3008	3008	1448	3008	3008	1448

Table VII Skewness of Firm Size Distribution

5.6 Exit Rates

Table VIII shows the impact of contracting frictions on exit rates, using data from Pakistan. The dependent variable is the exit rate for each size quartile of the industry-district-year. Columns (1), (2), and (3) are included to show the baseline exit rates. Column (4) shows the differential impact on the exit rates for industries that produce relationship-specific goods. We find that more severe contracting frictions are associated with lower exit rates.

5.7 Robustness

In our regressions on Indian plants, our main specification uses, as size, a plant's total sales. In Appendix B.5.1 we study how contracting frictions affect mean reversion and volatility among two subsets of plants: those that produce a single five-digit product and those that operate as a single-plant firm.[25] In both cases, the results are consistent with our baseline specification, that more severe contracting frictions lead to slower firm dynamics.

As discussed earlier, locations with congested courts may be different from other locations in a number of ways. It is possible that court congestion is correlated with some other local characteristic such as the level of development, that differentially affects firms that produce relationship-specific goods relative to those that produce standardized goods. We explore this by adding to our baseline regressions interactions of relationship specificity and the following local characteristics: log income per capita, a measure of trust, linguistic fragmentation, fragmentation by caste, and corruption. In Appendix B.5.2, we show that our main coefficients of interest change little when adding these additional controls.

Industries that produce relationship-specific goods are different from industries that produce standardized goods, e.g., in upstreamness or in capital intensity. It is possible that it is through

[25] Each plant is associated with an indicator of whether it is a standalone plant or member of a multi-plant firm.

	Dependent variable: P(exit)			
	(1)	(2)	(3)	(4)
Q1 Dummy	0.0738***	0.0717***		
	(0.0023)	(0.0057)		
Q2 Dummy	0.0255***	0.0208***	-0.0460***	-0.0469***
	(0.0018)	(0.0033)	(0.0013)	(0.0042)
Q3 Dummy	0.0131***	0.00979***	-0.0576***	-0.0567***
	(0.00099)	(0.0016)	(0.0016)	(0.0043)
Q4 Dummy	0.00800***	0.00677***	-0.0611***	-0.0586***
	(0.00071)	(0.0011)	(0.0018)	(0.0044)
Q1 × Relspec × AvgAgeCourts		0.00129		-0.00539*
		(0.0026)		(0.0025)
Q2 × Relspec × AvgAgeCourts		0.00299*		-0.00501**
		(0.0014)		(0.0019)
Q3 × Relspec × AvgAgeCourts		0.00221*		-0.00627***
		(0.00099)		(0.0016)
Q4 × Relspec × AvgAgeCourts		0.000871		-0.00755***
		(0.00087)		(0.0016)
Industry × Year FE			Yes	Yes
R^2	0.0525	0.0526	0.0460	0.0462
Observations	417711	411541	417698	411528

Standard errors in parentheses, clustered at the industry-region level.
$^{+}$ $p < 0.10$, * $p < 0.05$, ** $p < 0.01$

Table VIII Exit Rates by Size

one of those other industry characteristics that court congestion affects firm dynamics. To address this, we incorporate specifications in which we control for the interactions of court congestion with the following industry characteristics: capital intensity, industry wage premium, the share of contract workers, industry upstreamness, and tradability. As we report in Appendix B.5.3, the main coefficients of interest are insensitive to incorporating these additional controls.

6 Aggregate Productivity

In this section we return to the calibrated model to assess the impact of weak contract enforcement on aggregate productivity. In the face of weak formal enforcement, firms are less likely to encounter and switch to suppliers with lower cost, which reduces aggregate productivity. We can see this misallocation in two complementary ways: (i) the correlation between log cost and log size, and (ii) dispersion in size among those with the same cost.[26]

Table IX shows the correlation of cost and size among firms that produce relationship-specific inputs, using simulated data, for industries with and without frictions. The correlation is measured across all such firms in two ways, first by subtracting the industry mean from each observation, and

[26]There is a parallel between these two complementary measures of misallocation with the slope coefficient and R-squared of a regression as complementary measures of explanatory power.

second by subtracting the industry mean and dividing by the industry standard deviation. In either case, the correlation is weaker in the economy with more severe contracting frictions, indicating more severe misallocation.[27]

Model	Correlation (demeaned)	Correlation (normalized)
No friction	-0.281	-0.370
With friction	-0.260	-0.340

Table IX Correlation of Log Cost and Log Size

Note: This table shows the correlation of log cost and log size among firms that produce relationship-specific inputs. The correlation is measured across all such firms. In the column labeled "demeaned," we subtract from each firm's log cost and log size the respective industry mean. In the column labeled "normalized," we subtract the industry mean and divide by the industry standard deviation. Data from simulations of the calibrated model.

Figure 10 shows a second measure of misallocation, the dispersion in size among firms in the same cost decile. For almost every cost decile, there is more dispersion in size when contracting frictions are more severe.[28]

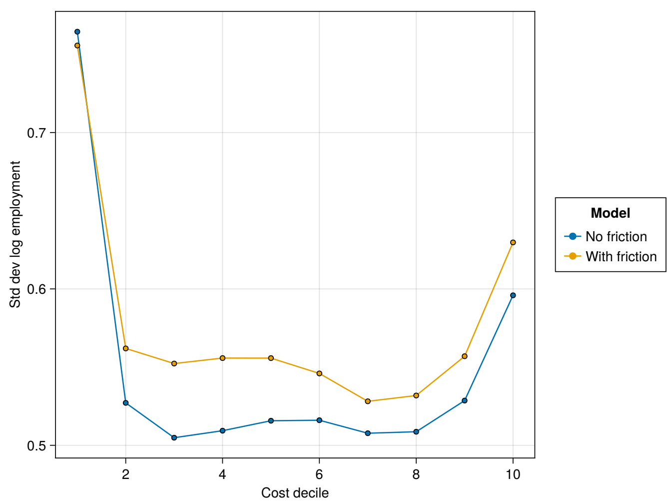


Figure 10 Dispersion in Size for each Cost Bin

Note: This figure shows the standard deviation of log employment among firms with the same cost, among firms in industries that produce relationship-specific industries. Data from simulations of the calibrated model.

Finally, Table X shows the implications for aggregate productivity. We conduct a counterfactual where we adjust κ (or likewise ϕ) to move from a situation where the average age of pending court cases is four years, to one where it is one year, as captured by the slope of 0.25 of the relationship duration to court congestion relationship (Table IV), holding the production function and demand

[27]Even without contracting frictions , there will be an imperfect correlation between log cost and size, because of the random arrival of customers and the random sizes of those customers.

[28]The exception is the lowest cost decile, in which dispersion in size in the economy with less severe contracting frictions is driven by the thicker right tail of the firm size distribution, as discussed in Section 4.

parameters (weights of industries in final demand) constant. As discussed in Section 2, the growth rate of aggregate productivity is invariant to long term relationships, as this depends only on the population growth rate (the model is one of semi-endogenous growth). Nevertheless, the contracting frictions reduce the *level* of aggregate productivity. In particular, by reducing the rate of reallocation of buyers across suppliers, court congestion in India's state with the most congested courts reduces aggregate productivity in the state by about 15% (16.2 log points) relative to the court congestion in the state with the least congested courts.[29] These dynamic losses from contracting frictions are roughly three times as large as the losses arising from static distortions in the form of transaction costs, estimated in the same context (Boehm and Oberfield, 2020).

	No friction	**With friction**
Mean income growth	0.015	0.015
Log real income difference	0.000	-0.162

Table X Aggregate Productivity

7 Conclusion

We present a model of firm dynamics where firms engage in vertical trading relationships. Firms continually draw new suppliers along with match-specific productivities and choose to switch to a new supplier when it is optimal to do so. The model generates moments that match the stylized empirical facts from the firm dynamics literature: firm volatility and exit probabilities decline with size, and growth rates exhibit fat tails. We apply the model to the study of contracting frictions in developing countries. Using data from India and Pakistan we show that when firms face contracting frictions with their suppliers they engage in long-term relationships, which in turn reduces firm volatility and the degree of mean reversion in size, and makes the tails of the firm size distribution thinner. We confirm these predictions in the model, and calibrate the model to match the data moments. Our model predicts that the dynamic cost of contract enforcement is large: moving from a location with the best observed contract enforcement to one with the worst observed contract enforcement entails a 15% reduction in aggregate productivity.

[29] Two notes: the magnitude of the loss in aggregate productivity scales one-for-one with $1/\beta$, and our calibration of β is based on Baqaee et al. (2023). Second, in the counterfactuals, we do not adjust the measure of firms that are created.

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A Data

A.1 Pakistani Court Data

Our measure of court congestion is, similarly to India, the average age of pending civil cases, which we construct from the 2011, 2020, and 2021 years of the "Judicial Statistics of Pakistan" reports (National Judicial Policy Making Committee, 2021). In contrast to India, we construct this from cases in district courts due to the lower number of provinces (the most closely corresponding administrative unit to Indian states in Pakistan). We also have no indication that parties are able to bypass district courts in Pakistan. We have data, on the aggregate provincial level, for Sindh, Khyber-Pakhtunkhwa, Balochistan, and Islamabad (ICT), and at the district level for the 37 districts of the Punjab. For Balochistan we use data from the 2021 Judicial Statistics of Pakistan because the 2020 numbers are incoherent.

The raw data contains the number of civil cases pending at the end of 2020 that were instituted in each year, between 2010 and 2020, and before 2010. From that we calculate the average age of pending cases, assuming that (i) institution of cases is uniformly distributed within the year; (ii) cases instituted up to the end of 2010 are, on average, 11.5 years old.

A.2 Pakistani VAT data

B Robustness and Further Results

B.1 Robustness of the size-volatility relationship

Tables XI replicates the analysis of the size-volatility relationship in the main text (Table II), but uses $((y_{t+1} - y_t)/0.5(y_{t+1} + y_t))$ to measure growth rather than $\Delta \log \text{Sales}_{t+1}$. This measure, popularized by Davis, Haltiwanger and Schuh (1998), incorporates observations in which a firm exits.

B.2 Exit Rates

Having very few customers is a strong predictor of exit. Figure 11 shows exit rate among firms with a given number of customers. Number of customers is a clear predictor of exit.

Table XII shows regressions of exit rates on combinations of fixed effects for number of buyers and fixed effects for sales ventiles. Number of buyers and sales have similar predictive power, as measured by R^2 (9% and 10%). Further, after including the number of customers, the additional predictive power from number of buyers is small, with R^2 rising from 9% to 11%.

The first column of Table XIII repeats these regressions on model generated data. Number of buyers is a good predictor of exit, although sales is not. That said, the final row shows that number of buyers is not a sufficient statistic; conditional on the number of buyers, sales helps predict exit, raising the R^2 marginally. The second column shows that when focusing on non-retail buyers, the

	Data					Simulation				
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
$\overline{\log(\text{Sales})}$	-0.119		-0.0644	-0.0832	-0.0816	-0.1854		-0.108	-0.1027	-0.1027
	(0.0015)		(0.002)	(0.0017)	(0.0018)	(0.0005)		(0.0007)	(0.0006)	(0.0006)
$\overline{\log(\text{Buyers})}$		-0.205	-0.132				-0.3865	-0.2464		
		(0.0025)	(0.0033)				(0.0011)	(0.0014)		
$\overline{\log(\text{HHI})}$				0.163	0.209				0.3571	0.3703
				(0.0044)	(0.0053)				(0.0013)	(0.009)
$\overline{\log(\text{HHI (weighted)})}$					-0.0495					-0.0133
					(0.0028)					(0.009)
Fixed Effects										
Industry	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Statistics										
R^2	0.27	0.286	0.315	0.308	0.311	0.733	0.736	0.7473	0.7637	0.7637
R^2-within	0.202	0.219	0.251	0.244	0.247	0.1746	0.1838	0.2188	0.2695	0.2695
Observations	24,784	24,784	24,784	24,784	24,124	543,869	543,869	543,869	543,869	543,869

Standard errors in parentheses.

Table XI Determinants of firm growth volatility

additional explanatory of power of sales rises. This is consistent with the idea that a firm would exit if its lone customer loses all of its customers, which is less likely to happen when its customer is large and has many buyers.

Table XII Exit rates by number of customers and size bin: Data

	Dependent variable: P(exit)			
	(1)	(2)	(3)	(4)
Constant	0.0878**	0.0879**	0.0878**	0.0879**
	(0.00039)	(0.00038)	(0.00038)	(0.00038)
Fixed Effects	Year	Year, #Buyers	Year, Sales vingtiles	Year, #Buyers, Sales vingtiles
R^2	0.0293	0.0889	0.0976	0.112
Observations	501828	501431	501828	501431

Standard errors in parentheses, clustered at the industry-region level.
$^+$ $p < 0.10$, * $p < 0.05$, ** $p < 0.01$

Note: Each column shows the results (notably the R^2) of a regression of exit dummies on a set of fixed effects.

Table XIII Exit rates by number of customers and size bin: Model

	Fixed effects			R^2	
	Year	#Buyers	Sales vingtile	All buyers	Non-retail buyers
1	✓			0.000	0.000
2	✓	✓		0.112	0.044
3	✓		✓	0.015	0.015
4	✓	✓	✓	0.116	0.055

Each column shows the results (notably the R^2) of a regression of exit dummies on a set of fixed effects.

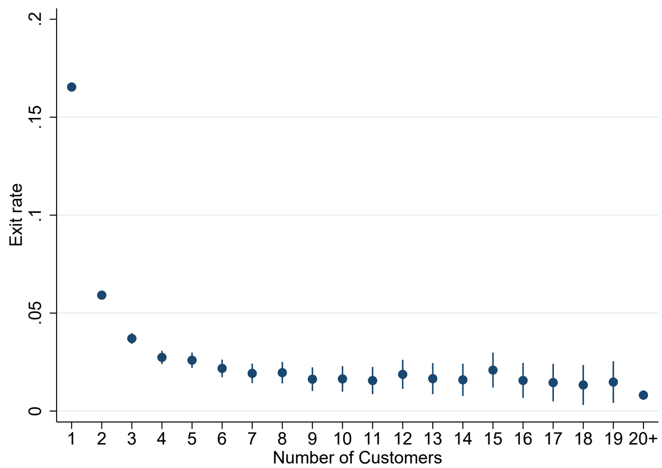

Figure 11 Exit Rate by Number of Customers

Note: This figure shows the exit rate among firms with a given number of customers in the Pakistan VAT data.

B.3 Age and Autocorrelation of Growth Rates

This section studies how the autocorrelation of growth rates varies with age in model-generated data, following the specification of Coad, Daunfeldt and Halvarsson (2018):

$$\text{Growth}_{it} \sim \log(\text{Sales}_{it}) + \text{Growth}_{it-1} + \text{Age}_t + \text{Growth}_{it-1} \times \text{Age}_t$$

The first two columns follow Coad, Daunfeldt and Halvarsson (2018) in using a median a median regression, with fixed effects are implemented by changing the dependent variable to $\hat{g}_{it} = \text{Growth}_{it} - \hat{\alpha}_{g(it)}$, where $\hat{\alpha}_{g(it)}$ is the fixed effect from a standard linear fixed effects regression. The third and fourth columns use OLS.

Bottazzi et al. (2011) and Coad, Daunfeldt and Halvarsson (2018) stress the importance of using median regressions because of the fat tailed nature of firm growth rates, and the fact that OLS disproportionately weights outliers. In the model, whether or not one focuses on outliers observations is important. A firm that gets an especially large growth rate because it is small and gains a large customer is especially likely to have a negative growth rate with a large magnitude if it loses that same customer. These dynamics are especially likely for firms with few customers, a category that disproportionately includes young firms. As a result, the OLS which disproportionately weights outliers yields a positive coefficient on the interaction of age and lagged growth, whereas the median regression favored by the literature because it which downweights outliers yields a negative coefficient.

	Median		Mean	
	(1)	(2)	(3)	(4)
Log sales	-0.5388	-0.5386	-0.6263	-0.6263
	(0.000)	(0.000)	(0.001)	(0.001)
Lag growth	0.0105	0.0140	-0.0550	-0.0559
	(0.000)	(0.000)	(0.000)	(0.001)
Age	-0.0005	-0.0005	0.0018	0.0018
	(0.000)	(0.000)	(0.000)	(0.000)
Lag growth × Age		-0.0003		0.0001
		(0.000)		(0.000)
Fixed effects	Industry	Industry	Industry	Industry

Table XIV Age and autocorrelation of firm growth rates

B.4 Results from Indian regression in Pakistan

The following tables we replicate the results from the baseline regressions that were done using data from India but with data from Pakistan. Note, in particular, the following key differences between the ASI data in India, and the VAT data from Pakistan:

- The data from Pakistan are VAT data, and include only information on sales values and sales relationships, but no data on primary factors (employment, wage bill, investment etc) or sales by product. The unit of observation in the Pakistani data is the firm, which is identified by its tax IDs (NTN and STRN), whereas the unit of observation in the Indian data is the plant.
- Industry codes in the Pakistani data are available at the 2-digit and 4-digit level, but are very incomplete at the 4-digit level. We therefore use 2-digit industry codes.
- Geographic information in the Pakistani data comes from an address string, which is available in 2011 and 2019. We use the geocoding from Balboni, Boehm and Waseem (2024) and prefer 2011 addresses over 2019, whenever available. We mostly use districts as our level of spatial aggregation.

Table XV shows that mean reversion among firms that produce relationship-specific goods weakens more with court congestion than mean reversion among firms that produce standardized goods. Table XVI shows that the right tail of the size distribution among firms that produce relationship-specific goods grows thinner with court congestion than the right tail among firms that produce standardized goods.

B.5 Robustness checks for the baseline regressions on Indian data

B.5.1 Single Product Plants and Standalone Plants

The following tables show robustness checks for the baseline regressions that use Indian ASI data. Table XIX shows results using only single-product plants; Table XX shows results using only plants that have no sister plants belonging to the same firm (i.e. single-plant firms). Note that the latter

	Dependent variable: Change in log Sales		
	(1)	(2)	(3)
$\log \text{Sales}_{t-1}$	-0.146**	-0.163**	-0.163**
	(0.0051)	(0.010)	(0.011)
$\log \text{Sales}_{t-1} \times$ Age civ. cases × relspec		0.0114^{+}	0.0128*
		(0.0060)	(0.0062)
Firm × 2-digit Industry FE	Yes	Yes	Yes
District FE	Yes	Yes	
Year FE	Yes	Yes	
Age FE			Yes
Industry × District × Year FE			Yes
R^2	0.218	0.218	0.249
Observations	205351	205254	201931

Standard errors in parentheses, clustered at the district × industry level.

Table XV Mean Reversion: Pakistan

	Dependent variable: Skewness of log Sales		
	(1)	(2)	(3)
Relspec x Court Congestion	-0.914	-1.053^{+}	-1.465^{+}
	(0.593)	(0.562)	(0.831)
District FE	Yes	Yes	Yes
2-digit Industry FE	Yes	Yes	Yes
Statistic	25-75	25-90	50-90
R^2	0.424	0.598	0.547
Observations	935	688	688

$^{+}$ $p < 0.10$, * $p < 0.05$, ** $p < 0.01$

Table XVI Firm Size Distribution: Pakistan

also constrains the data to the interval 2001–2010 since this information is only reliably available for that time period.

B.5.2 State Characteristics

B.5.3 Industry Characteristics

	Dependent variable: $\sigma(\Delta \log \text{Sales})_{d\omega}$			
	(1)	(2)	(3)	(4)
Avg age of civil cases × Rel. spec.	-0.0254*	-0.0258*	-0.0446*	-0.0439*
	(0.012)	(0.012)	(0.021)	(0.020)
$\overline{(\Delta \log \text{Sales})_{d\omega}}$		-0.288**		-0.288**
		(0.022)		(0.022)
State FE	Yes	Yes	Yes	Yes
5-digit Industry FE	Yes	Yes	Yes	Yes
Estimator	OLS	OLS	IV	IV
R^2	0.284	0.305	0.000332	0.0300
Observations	7028	7028	7028	7028

Regression at the state × industry level. Only state-industry cells with more than 3 observations used.

Table XVII Lower variance of sales growth when frictions are large: single-product plants only

	Dependent variable: $\sigma(\Delta \log \text{Sales})_{d\omega}$			
	(1)	(2)	(3)	(4)
Avg age of civil cases × Rel. spec.	-0.00253	-0.00336	-0.0229	-0.0236
	(0.012)	(0.012)	(0.021)	(0.021)
$\overline{(\Delta \log \text{Sales})_{d\omega}}$		-0.160**		-0.161**
		(0.026)		(0.026)
State FE	Yes	Yes	Yes	Yes
5-digit Industry FE	Yes	Yes	Yes	Yes
Estimator	OLS	OLS	IV	IV
R^2	0.304	0.311	-0.000736	0.00862
Observations	5002	5002	5002	5002

Regression at the state × industry level. Only state-industry cells with more than 3 observations used.

Table XVIII Lower variance of sales growth when frictions are large: single-plant firms only

	Dependent variable: Change in log Sales					
	(1)	(2)	(3)	(4)	(5)	(6)
$\log \text{Sales}_{t-1}$	-0.428**	-0.461**	-0.546**	-0.422**	-0.488**	-0.624**
	(0.013)	(0.037)	(0.062)	(0.014)	(0.042)	(0.065)
$\log \text{Sales}_{t-1}$× Age civ. cases × relspec	0.00447	0.0219+	0.0182	0.00165	0.0339+	0.0536+
	(0.0043)	(0.013)	(0.020)	(0.0051)	(0.017)	(0.029)
Plant × 5-digit Industry FE	Yes	Yes	Yes	Yes	Yes	Yes
State FE	Yes			Yes		
Year × Previous Year FE	Yes			Yes		
Age FE		Yes	Yes		Yes	Yes
Industry × District × Year FE		Yes			Yes	
Industry × District ×$(t, t-1)$ FE			Yes			Yes
Method	OLS	OLS	OLS	IV	IV	IV
R^2	0.481	0.629	0.655	0.268	0.248	0.260
Observations	110279	36767	24528	110279	36767	24528

Standard errors in parentheses, clustered at the state × industry level.

Table XIX Mean Reversion: Single-product plants only

	Dependent variable: Change in log Sales					
	(1)	(2)	(3)	(4)	(5)	(6)
$\log \text{Sales}_{t-1}$	-0.466**	-0.515**	-0.696**	-0.468**	-0.535**	-0.742**
	(0.017)	(0.034)	(0.042)	(0.019)	(0.037)	(0.039)
$\log \text{Sales}_{t-1} \times$ Age civ. cases $\times$ relspec	0.0140*	0.0336**	0.0471**	0.0148*	0.0445**	0.0706**
	(0.0064)	(0.013)	(0.015)	(0.0073)	(0.017)	(0.022)
Plant × 5-digit Industry FE	Yes	Yes	Yes	Yes	Yes	Yes
State FE	Yes			Yes		
Year × Previous Year FE	Yes			Yes		
Age FE		Yes	Yes		Yes	Yes
Industry × District × Year FE		Yes			Yes	
Industry × District $\times (t, t-1)$ FE			Yes			Yes
Method	OLS	OLS	OLS	IV	IV	IV
R^2	0.485	0.649	0.685	0.271	0.266	0.302
Observations	74248	29634	19030	74248	29634	19030

Standard errors in parentheses, clustered at the state × industry level.

Table XX Mean Reversion: Single-plant firms only

	Dependent variable: $\sigma(\Delta \log \text{Sales})_{d\omega}$			
	(1)	(2)	(3)	(4)
Avg age of civil cases × Rel. spec.	-0.0226*	-0.0239*	-0.0502**	-0.0513**
	(0.010)	(0.010)	(0.015)	(0.015)
Log GDPC × Rel. Spec.	0.0147	0.0155	0.00355	0.00440
	(0.017)	(0.017)	(0.018)	(0.018)
Trust × Rel. Spec.	0.173+	0.174+	0.205*	0.206*
	(0.095)	(0.094)	(0.096)	(0.095)
Language HHI × Rel. Spec.	0.0661	0.0593	0.0593	0.0526
	(0.090)	(0.090)	(0.090)	(0.090)
Caste HHI × Rel. Spec.	-0.190	-0.199+	-0.226+	-0.234*
	(0.12)	(0.12)	(0.12)	(0.12)
Corruption × Rel. Spec.	-0.525+	-0.477	-0.612*	-0.563+
	(0.30)	(0.30)	(0.31)	(0.30)
$\overline{(\Delta \log \text{Sales})_{d\omega}}$		-0.238**		-0.239**
		(0.027)		(0.027)
State FE	Yes	Yes	Yes	Yes
5-digit Industry FE	Yes	Yes	Yes	Yes
Estimator	OLS	OLS	IV	IV
R^2	0.310	0.321	0.00157	0.0171
Observations	5909	5909	5909	5909

Regression at the state × industry level. Only state-industry cells with more than 5 observations used.

Table XXI Variance of sales growth: State characteristics robustness

	Dependent variable: Change in log Sales					
	(1)	(2)	(3)	(4)	(5)	(6)
$\log \text{Sales}_{t-1}$	-0.387**	-0.425**	-0.572**	-0.388**	-0.427**	-0.575**
	(0.015)	(0.033)	(0.045)	(0.015)	(0.033)	(0.045)
$\log \text{Sales}_{t-1} \times$ Age civ. cases $\times$ relspec	0.0197**	0.0137	-0.0142	0.0503**	0.0641**	0.0899**
	(0.0067)	(0.015)	(0.027)	(0.0094)	(0.022)	(0.034)
$\log \text{Sales}_{t-1} \times$ Log GDPC $\times$ Rel. Spec.	-0.0114*	-0.00331	0.0132	-0.0162**	-0.0150	-0.0115
	(0.0055)	(0.013)	(0.021)	(0.0060)	(0.016)	(0.027)
$\log \text{Sales}_{t-1} \times$ Trust $\times$ Rel. Spec.	0.0324	0.119	0.150	-0.0270	0.0857	0.121
	(0.068)	(0.18)	(0.30)	(0.071)	(0.19)	(0.34)
$\log \text{Sales}_{t-1} \times$ Language HHI $\times$ Rel. Spec.	0.0472	-0.0315	-0.0522	0.0130	-0.0436	-0.0633
	(0.053)	(0.12)	(0.21)	(0.056)	(0.13)	(0.27)
$\log \text{Sales}_{t-1} \times$ Caste HHI $\times$ Rel. Spec.	0.132	0.304	-0.0914	0.167^{+}	0.359	-0.0190
	(0.088)	(0.20)	(0.35)	(0.096)	(0.23)	(0.41)
$\log \text{Sales}_{t-1} \times$ Corruption $\times$ Rel. Spec.	0.426**	0.136	0.230	0.482**	0.212	0.291
	(0.16)	(0.28)	(0.53)	(0.17)	(0.34)	(0.64)
Plant $\times$ 5-digit Industry FE	Yes	Yes	Yes	Yes	Yes	Yes
State FE	Yes			Yes		
Year $\times$ Previous Year FE	Yes			Yes		
Age FE		Yes	Yes		Yes	Yes
Industry $\times$ District $\times$ Year FE		Yes			Yes	
Industry $\times$ District $\times (t, t-1)$ FE			Yes			Yes
Method	OLS	OLS	OLS	IV	IV	IV
R^2	0.446	0.628	0.660	0.250	0.247	0.270
Observations	159914	65943	43946	159914	65943	43946

Standard errors in parentheses, clustered at the state $\times$ industry level.

Table XXII Mean Reversion: State characteristics robustness

	Dependent variable: Skewness of log Sales					
	(1)	(2)	(3)	(4)	(5)	(6)
Relspec x Court Congestion	-0.415*	-0.825**	-0.894**	-0.638*	-1.020*	-0.726
	(0.185)	(0.314)	(0.325)	(0.280)	(0.476)	(0.473)
Log GDPC × Rel. Spec.	-0.449	-1.423*	-0.984	-0.565	-1.524*	-0.903
	(0.351)	(0.597)	(0.674)	(0.367)	(0.625)	(0.695)
Trust × Rel. Spec.	-0.118	-3.063	-3.447	0.0153	-2.947	-3.571
	(1.680)	(2.859)	(2.993)	(1.686)	(2.868)	(3.005)
Language HHI × Rel. Spec.	-0.985	-3.004	-2.652	-0.873	-2.906	-2.736
	(1.268)	(2.158)	(2.338)	(1.273)	(2.166)	(2.344)
Caste HHI × Rel. Spec.	-3.594+	-3.276	-4.020	-3.841+	-3.490	-3.762
	(1.971)	(3.354)	(3.696)	(1.985)	(3.378)	(3.734)
R^2	0.564	0.468	0.564	0.004	0.006	0.013
State FE	Yes	Yes	Yes	Yes	Yes	Yes
5-digit Industry FE	Yes	Yes	Yes	Yes	Yes	Yes
Estimator	OLS	OLS	OLS	IV	IV	IV
Statistic	25-75	50-75	50-90	25-75	50-75	50-90
Observations	2414	2414	1146	2414	2414	1146

Table XXIII Skewness of Firm Size Distribution: State characteristics robustness

	Dependent variable: $\sigma(\Delta \log \text{Sales})_{d\omega}$			
	(1)	(2)	(3)	(4)
Avg age of civil cases × Rel. spec.	-0.0174+	-0.0182+	-0.0424*	-0.0402*
	(0.0096)	(0.0095)	(0.018)	(0.018)
Capital Intensity * Avg. age of cases	-0.0611	-0.0553	-0.0772	-0.0694
	(0.13)	(0.13)	(0.13)	(0.13)
Ind. Wage Premium * Avg. age of cases	0.00704*	0.00708**	0.00748**	0.00746**
	(0.0028)	(0.0027)	(0.0028)	(0.0028)
Ind. Contract Worker Share * Avg. age of cases	0.0629	0.0588	0.0591	0.0554
	(0.061)	(0.061)	(0.062)	(0.061)
Upstreamness * Avg. age of cases	0.00211	0.00237	-0.00200	-0.00125
	(0.0049)	(0.0048)	(0.0055)	(0.0054)
Tradability * Avg. age of cases	-0.0000534	0.0000250	-0.000163	-0.0000718
	(0.0013)	(0.0012)	(0.0013)	(0.0012)
$\overline{(\Delta \log \text{Sales})_{d\omega}}$		-0.273**		-0.273**
		(0.024)		(0.024)
State FE	Yes	Yes	Yes	Yes
5-digit Industry FE	Yes	Yes	Yes	Yes
Estimator	OLS	OLS	IV	IV
R^2	0.287	0.302	0.000749	0.0218
Observations	7562	7562	7562	7562

Regression at the state × industry level. Only state-industry cells with more than 5 observations used.

Table XXIV Variance of sales growth: Industry characteristics robustness

	Dependent variable: Change in log Sales					
	(1)	(2)	(3)	(4)	(5)	(6)
$\log \text{Sales}_{t-1}$	-0.390**	-0.374**	-0.426**	-0.388**	-0.375**	-0.435**
	(0.014)	(0.029)	(0.054)	(0.015)	(0.029)	(0.053)
$\log \text{Sales}_{t-1} \times$ Age civ. cases $\times$ relspec	0.00749^{+}	0.0302**	0.0332*	0.00621	0.0312**	0.0390*
	(0.0040)	(0.0099)	(0.014)	(0.0044)	(0.011)	(0.016)
$\log \text{Sales}_{t-1} \times$ Capital Intensity $\times$ Avg. age of cases	0.000958	-0.0881	-0.355	0.00223	-0.0868	-0.332
	(0.059)	(0.19)	(0.28)	(0.059)	(0.19)	(0.27)
$\log \text{Sales}_{t-1} \times$ Ind. Wage Premium $\times$ Avg. age of cases	0.0000719	-0.00627	-0.00518	0.000171	-0.00640	-0.00593
	(0.0011)	(0.0040)	(0.0061)	(0.0011)	(0.0041)	(0.0060)
$\log \text{Sales}_{t-1} \times$ Ind. Contract Worker Share $\times$ Avg. age	0.000125	0.0206	-0.00330	-0.000260	0.0208	-0.00557
	(0.023)	(0.036)	(0.050)	(0.023)	(0.036)	(0.049)
$\log \text{Sales}_{t-1} \times$ Upstreamness $\times$ Avg. age of cases	-0.000230	0.00104	0.00258	-0.000276	0.00112	0.00310
	(0.00090)	(0.0023)	(0.0038)	(0.00091)	(0.0023)	(0.0038)
$\log \text{Sales}_{t-1} \times$ Tradability $\times$ Avg. age of cases	-0.000671	-0.00121	-0.00309^{+}	-0.000656	-0.00124	-0.00320^{+}
	(0.00052)	(0.0013)	(0.0018)	(0.00052)	(0.0013)	(0.0018)
Plant $\times$ 5-digit Industry FE	Yes	Yes	Yes	Yes	Yes	Yes
State FE	Yes			Yes		
Year $\times$ Previous Year FE	Yes			Yes		
Age FE		Yes	Yes		Yes	Yes
Industry $\times$ District $\times$ Year FE		Yes			Yes	
Industry $\times$ District $\times (t, t-1)$ FE			Yes			Yes
Method	OLS	OLS	OLS	IV	IV	IV
R^2	0.457	0.635	0.668	0.257	0.251	0.281
Observations	203563	77828	51222	203563	77828	51222

Standard errors in parentheses, clustered at the state $\times$ industry level.

Table XXV Mean Reversion: Industry characteristics robustness

	Dependent variable: Skewness of log Sales					
	(1)	(2)	(3)	(4)	(5)	(6)
Relspec x Court Congestion	-0.356^{+}	-0.682*	-0.999**	-0.674	-1.412*	-1.124^{+}
	(0.183)	(0.314)	(0.320)	(0.415)	(0.711)	(0.678)
Capital Intensity * Avg. age of cases	1.337	-1.581	-3.547	1.178	-1.943	-3.665
	(2.818)	(4.824)	(5.435)	(2.826)	(4.840)	(5.464)
Ind. Wage Premium * Avg. age of cases	-0.0325	0.0232	0.118^{+}	-0.0233	0.0442	0.122^{+}
	(0.0430)	(0.0737)	(0.0655)	(0.0444)	(0.0760)	(0.0681)
Ind. Contract Worker Share * Avg. age of cases	-0.112	-3.154	-1.051	-0.281	-3.541	-1.193
	(1.307)	(2.237)	(2.759)	(1.323)	(2.265)	(2.842)
Upstreamness * Avg. age of cases	-0.0176	-0.0294	-0.153	-0.0702	-0.150	-0.169
	(0.0916)	(0.157)	(0.149)	(0.110)	(0.189)	(0.167)
Tradability * Avg. age of cases	0.00650	0.0942^{+}	0.0293	0.00388	0.0883^{+}	0.0284
	(0.0310)	(0.0530)	(0.0608)	(0.0311)	(0.0533)	(0.0610)
R^2	0.540	0.436	0.556	0.001	0.002	0.011
State FE	Yes	Yes	Yes	Yes	Yes	Yes
5-digit Industry FE	Yes	Yes	Yes	Yes	Yes	Yes
Estimator	OLS	OLS	OLS	IV	IV	IV
Statistic	25-75	50-75	50-90	25-75	50-75	50-90
Observations	3008	3008	1448	3008	3008	1448

Table XXVI Skewness of Firm Size Distribution: Industry characteristics robustness

C Numerical simulations

C.1 Calibration

C.1.1 Population growth

The employment share of young firms is greater with a high population growth rate. Hsieh and Klenow (2014) give statistics on the employment share of plants by age in 2010-11 from the ASI and NSS in Figure III. But they also show that cohort sizes are larger post 1997. They also include a calculation for steady-state employment shares based on measured employment growth and exit rates assuming that all cohorts are equally sized in Figure VI.

C.2 Input Output Tables

To facilitate the numerical simulation, we make some modifications to the input output tables so that it is model consistent and more sparse. We first remove some links and industries.

We use the following notation: Expenditure between a buying and supplying industries, $E(b, s)$, is the total expenditure by single product plants in b on s. Production, $P(\omega)$ is the total ex-factory value of produced goods. Final consumption, $C(\omega)$, is production less total ASI expenditure on the product.

1. Only manufacturing products are included.
2. **Important links:** links that are unimportant for both supply and demand are removed to increase sparsity.
   - (Supply) For each purchasing industry b, let M be the set of supply industries. The supply-important pairs (b, s) are those with s in the smallest set $S \subseteq M$ such that

$$\sum_{s \in S} E(b, s) \geq 0.99 \times \sum_{s \in M} E(b, s)$$

Table XXVII Hsieh & Klenow employment shares

Age group	Employment share (2010-11)	Employment share (SS)
< 5	0.26	0.18
$5 - 9$	0.22	0.15
$10 - 14$	0.18	0.12
$15 - 19$	0.10	0.10
$20 - 24$	0.08	0.08
$25 - 29$	0.05	0.06
$30 - 34$	0.04	0.04
$35 - 39$	0.02	0.04
≥ 40	0.05	0.23

- (Demand) For each supplying industry s, let M be the set of purchasing industries. The demand-important pairs (b, s) are those with b in the smallest set $B \subseteq M$ such that

$$\sum_{b \in B} E(b, s) \geq 0.99 \times \sum_{b \in M} E(b, s)$$

- Expenditure on pairs which are neither supply-important nor demand-important is set to zero.

3. **Slow industries:** For some industries, calculating the firms' costs is very slow. These are industries with either very high expenditure share on manufacturing inputs and more generally those associated with large eigenvalues of the IO matrix.

   - We remove industries with total expenditure on manufacturing outputs greater than 95% of the ex-factory value of production.
   - We remove the industries associated with eigenvalues of the IO matrix above 0.5.

4. **Industries with no suppliers:** We remove industries which do not purchase inputs from any included industries.

5. **No path to final consumption:** We remove any industries with zero direct final consumption and zero (potentially indirect) customers with strictly positive final consumption.

Then given the remaining industries and buyer-supplier links, we reconstruct consistent IO tables as follows:

- We assume any purchases from industries or through links that are not in our final sample are spent on primary inputs.

- We assume any sales to industries or through links that are not in our sample are sales to retailers.

- $\alpha_{b,s} = \frac{E(b,s)}{P(b)}$

- $\alpha^H_\omega = \max\{0, C(\omega)\}$ (normalized to sum to 1 across industries)

- Given the IO matrix A, the fraction of firms in each industry ρ_ω is set to equalize the average value of output per firm across industries.

C.3 Simulation Procedure

We simulate the model using a discrete approximation using a finite number of firms.

In time step Δ, Given an arrival rate ϕ of techniques that dominate one's current supplier,

At each point in time, we have recorded for each firm the identity of each supplier and the match specific productivity. We compute each firm's cost as a fixed point.

If the time step is small enough, it is likely there is at most one new technique that dominates the existing technique. Among all techniques that arrive, the identity of the supplier is random. However, among techniques that are larger than any threshold, suppliers that have a low cost are over-represented. In particular, among techniques that beat the current supplier, the probability that it comes from supplier j is

$$\frac{c_j^{-\beta}}{\sum_{\tilde{j}\in J_{\hat{\omega}}} c_{\tilde{j}}^{-\beta}} \tag{7}$$

Thus we assume that in each time step, a new, better supplier arrives with probability $\Delta\phi$, and if it does arrive, the reduction in cost x is drawn from a Pareto with shape parameter β and the identity of the supplier is randomly drawn according to (7).

In practice, we use a time step Δ of one year. Our simulations are insensitive to varying the length of the time step.

We simulate the model for 150 years. We discard the first 100 years, as by this time the model has converged sufficiently to its balanced growth path, and the statistics we compute have stabilized. We then use the remaining 50 years to collect outcomes.

D Proofs

D.1 Static Equilibrium

We derive results here for the static equilibrium of the full multi-industry model. The simple model is just a special case.

D.1.1 Necessary Conditions for Pairwise Stability

Claim 1 *Fix a feasible contracting arrangement. The cost-minimizing choice of labor for firm b industry ω satisfies*

$$wl_b = \alpha_{\omega l} c_b y_b \tag{8}$$

and its marginal cost is

$$c_b = w\left(\frac{y_b}{A_\omega \prod_{\hat{\omega}}\left(z_{b,\mathfrak{s}(b,\hat{\omega})}x_{b,\mathfrak{s}(b,\hat{\omega})}\right)^{\alpha_{\omega\hat{\omega}}}}\right)^{1/\alpha_{\omega l}} \frac{1}{\alpha_{\omega l}}\frac{1}{y_b} \tag{9}$$

For any retailer, $p_r = \frac{\varepsilon}{\varepsilon-1}c_r$. and profit cannot be negative for any firm, $\pi_j \geq 0$, $\forall j$.

Proof. Given the contracting arrangement, if $y_b = 0$ (because either b is a producing firm and the contracting arrangement dictates $y_b = 0$, or because b is a retailer and b finds it optimal to choose $y_b = 0$), then it is optimal for b to choose $l_b = 0$.

Suppose instead that $y_{\omega b} > 0$. Since the contracting arrangement is feasible, it must be that each $x_{\omega,\mathfrak{s}(b,\hat{\omega})} > 0$. Given the contracting arrangement, firm b's cost minimization problem is to minimize labor subject to its production function. Firm b's total cost (including transfers to its suppliers) is

$$C_b(y) = \min_{l_b} wl_b + \sum_{\hat{\omega}} T_{b,\mathfrak{s}(b,\hat{\omega})}$$

subject to

$$y \leq A_\omega l_b^{\alpha_{\omega l}} \prod_{\hat{\omega}} \left(z_{b,\mathfrak{s}(b,\hat{\omega})} x_{b,\mathfrak{s}(b,\hat{\omega})}\right)^{\alpha_{\omega\hat{\omega}}}$$

Eliminating l_b gives

$$C_b(y) = w \left(\frac{y}{A_\omega \prod_{\hat{\omega}} \left(z_{b,\mathfrak{s}(b,\hat{\omega})} x_{b,\mathfrak{s}(b,\hat{\omega})}\right)^{\alpha_{\omega\hat{\omega}}}} \right)^{1/\alpha_{\omega l}} + \sum_{\hat{\omega}} T_{b,\mathfrak{s}(b,\hat{\omega})}$$

Marginal cost is then

$$c_b = C_b'(y) = w \left(\frac{y}{A_\omega \prod_{\hat{\omega}} \left(z_{b,\mathfrak{s}(b,\hat{\omega})} x_{b,\mathfrak{s}(b,\hat{\omega})}\right)^{\alpha_{\omega\hat{\omega}}}} \right)^{1/\alpha_{\omega l}} \frac{1}{\alpha_{\omega l}} \frac{1}{y}$$

This can be rearranged as

$$\alpha_{\omega l} c_b y_b = w \left(\frac{y_b}{A_\omega \prod_{\hat{\omega}} \left(z_{b,\mathfrak{s}(b,\hat{\omega})} x_{b\mathfrak{s}(b,\hat{\omega})}\right)^{\alpha_{\omega\hat{\omega}}}} \right)^{1/\alpha_{\omega l}} = wl_b$$

Retailer r's choice of price satisfies

$$\max_{p_r, y_r} p_r y_r - C_r(y_r) \text{ subject to } y_r \leq Y P^\varepsilon p_r^{-\varepsilon}$$

The first order conditions imply $p_r = \frac{\varepsilon}{\varepsilon - 1} C_r'(y_r) = \frac{\varepsilon}{\varepsilon-1} c_r$.

Finally, if $\pi_j < 0$, then j could improve its payoff by dropping all contracts.

■

Claim 2 *In any pairwise equilibrium, for each buyer-supplier pair b and s in respective industries ω and $\hat{\omega}$,*

$$c_s x_{bs} = \alpha_{\omega\hat{\omega}} c_b y_b \tag{10}$$

Proof.

Consider a feasible contracting arrangement. Given the contracting arrangement, let y_b be the output that b produces; if b is a firm, y_b is determined directly by the arrangement, and if b is a retailer, then y_b is a choice. In any case, we will show that given the arrangement and, if b is a retailer, the choice of y_b, the the pair b and its supplier s can increase their bilateral surplus if (10) does not hold.

First, suppose that $y_b = 0$. Pairwise stability requires that $x_{bs} = 0$, because otherwise the supplier could conserve on labor. Suppose otherwise, i.e. that $y_b > 0$. For the contracting arrangement to be feasible, it must be that all of the buyer's suppliers and all of the supplier's suppliers provide strictly positive input quantities.

Consider the supplier of good $\hat{\omega}$, s. Let $y_{s,-b}$ denote the quantity that the supplier s must provide to all customers aside from b. Thus the supplier's total output is $y_s = x_{bs} + y_{\omega s,-b}$. Given the supplier's production function, its expenditure on labor is

$$wl_s = w\left(\frac{x_{bs} + y_{s,-b}}{A_{\hat{\omega}} \prod_{\tilde{\omega}} \left[z_{s,\mathfrak{s}(s,\tilde{\omega})} x_{s,\mathfrak{s}(s,\tilde{\omega})}\right]^{\alpha_{\hat{\omega}\tilde{\omega}}}}\right)^{\frac{1}{\alpha_{\hat{\omega}l}}}$$

The sum of the expenditure on labor used by s and b is

$$w\left(\frac{x_{bs} + y_{s,-b}}{A_{\hat{\omega}} \prod_{\tilde{\omega}} \left[z_{s,\mathfrak{s}(s,\tilde{\omega})} x_{s,\mathfrak{s}(s,\tilde{\omega})}\right]^{\alpha_{\hat{\omega}\tilde{\omega}}}}\right)^{\frac{1}{\alpha_{\hat{\omega}l}}} + w\left(\frac{y_b}{A_{\omega} \prod_{\omega'} \left(z_{b,\mathfrak{s}(b,\omega')} x_{b,\mathfrak{s}(b,\omega')}\right)^{\alpha_{\omega\omega'}}}\right)^{\frac{1}{\alpha_{\omega l}}}$$

This is convex in x_{bs}, and the global minimum satisfies

$$w\frac{1}{\alpha_{\hat{\omega}l}}\frac{1}{x_{bs} + y_{s,-b}}\left(\frac{x_{bs} + y_{s,-b}}{A_{\hat{\omega}} \prod_{\tilde{\omega}} \left[z_{s,\mathfrak{s}(s,\tilde{\omega})} x_{s,\mathfrak{s}(s,\tilde{\omega})}\right]^{\alpha_{\hat{\omega}\tilde{\omega}}}}\right)^{\frac{1}{\alpha_{\hat{\omega}l}}} = \frac{\alpha_{\omega\hat{\omega}}}{\alpha_{\omega l}}\frac{1}{x_{bs}}w\left(\frac{y_b}{A_{\omega} \prod_{\omega'} \left(z_{b,\mathfrak{s}(b,\omega')} x_{b,\mathfrak{s}(b,\omega')}\right)^{\alpha_{\omega\omega'}}}\right)^{\frac{1}{\alpha_{\omega l}}}$$

using (9) for each firm gives

$$c_s = \frac{\alpha_{\omega\hat{\omega}}}{x_{bs}} y_b c_b$$

Thus, unless the marginal costs satisfy this condition, there is a profitable deviation in which the pair could make the same output using less labor. Transfers can then be adjusted so that both firms are better off. ■

Claim 3 *In any pairwise equilibrium,*

$$c_b = w^{1-\alpha_{\omega l}} \prod_{\hat{\omega}\in\Omega} \left(\frac{c_{\mathfrak{s}(b,\hat{\omega})}}{z_{b,\mathfrak{s}(b,\hat{\omega})}}\right)^{\alpha_{\omega\hat{\omega}}}$$

Proof. This follows from (8) for b, (10) for b and each of its suppliers, and b's production function. ■

For any buyer seller pair, define

$$\tau_{bs} \equiv T_{bs} - c_s x_{bs} \ .$$

A firm's profit is

$$\pi_j = \sum_{b\in\mathcal{B}_j} T_{bj} - wl_j - \sum_{\hat{\omega}} T_{j,\mathfrak{s}(j,\hat{\omega})} \qquad \text{(producer)}$$

$$\pi_j = p_j y_j - wl_j - \sum_{\hat{\omega}} T_{j,\mathfrak{s}(j,\hat{\omega})} \qquad \text{(retailer)}$$

Claim 4 *In any pairwise stable equilibrium,*

$$\begin{aligned}\pi_j &= \sum_{b\in\mathcal{B}_j}\tau_{bj}-\sum_{\hat\omega}\tau_{j,\mathfrak{s}(j,\hat\omega)} && \textit{(producer)}\\ \pi_j &= (p_j-c_j)YP^{\varepsilon}p_j^{-\varepsilon}-\sum_{\hat\omega}\tau_{j,\mathfrak{s}(j,\hat\omega)} && \textit{(retailer)}\end{aligned}$$

Proof. For a producing firm, profit is

$$\begin{aligned}\pi_j &= \sum_{b\in\mathcal{B}_j}T_{bj}-wl_j-\sum_{\hat\omega}T_{j,\mathfrak{s}(j,\hat\omega)}\\ &= \sum_{b\in\mathcal{B}_j}(\tau_{bj}+c_jx_{bj})-wl_j-\sum_{\hat\omega}\left(\tau_{j,\mathfrak{s}(j,\hat\omega)}+c_{\mathfrak{s}(j,\hat\omega)}x_{j\mathfrak{s}(j,\hat\omega)}\right)\end{aligned}$$

Using $y_j=\sum_{b\in\mathcal{B}_j}x_{bj}$, $wl_j=\alpha_{\omega l}c_jy_j$, and $c_{\mathfrak{s}(j,\hat\omega)}x_{j,\mathfrak{s}(j,\hat\omega)}=\alpha_{\omega\hat\omega}c_jy_j$, this is

$$\pi_j=\sum_{b\in\mathcal{B}_j}\tau_{bj}-\sum_{\hat\omega}\tau_{j,\mathfrak{s}(j,\hat\omega)}$$

For a retailer, profit is

$$\begin{aligned}\pi_j &= p_jy_j-wl_j-\sum_{\hat\omega}T_{j,\mathfrak{s}(j,\hat\omega)}\\ &= (p_j-c_j)\,y_j+c_jy_j-wl_j-\sum_{\hat\omega}\left(\tau_{j,\mathfrak{s}(j,\hat\omega)}+c_{\mathfrak{s}(j,\hat\omega)}x_{j,\mathfrak{s}(j,\hat\omega)}\right)\end{aligned}$$

Using $wl_j=\alpha_{\omega l}c_jy_j$ and $c_{\mathfrak{s}(j,\hat\omega)}x_{j,\mathfrak{s}(j,\hat\omega)}=\alpha_{\omega\hat\omega}c_jy_j$, this is

$$\pi_j=(p_j-c_j)\,y_j-\sum_{\hat\omega}\tau_{j,\mathfrak{s}(j,\hat\omega)}$$

■

D.1.2 Feasibility and a Supply Tree Representation

Consider firm j in industry ω. We first characterize j's supply tree. It will be useful to visualize this supply tree as a graph with firm j at the root. There is set of edges connecting j to each of its suppliers, $\{\mathfrak{s}(j,\hat\omega)\}_{\hat\omega}$, and for each of those suppliers, a set of edges connecting to each of the suppliers' suppliers, etc.

The structure of the supply tree is the same for all firms in the supply tree. For firms in ω, let Ψ_ω represent the set of nodes with representative element ψ.

In any supply tree, there is a partial ordering $\geq$ such that $\psi\geq\psi'$ if ψ is weakly upstream from ψ' (that is, the path from the root to ψ contains the path from the root to ψ').

Let $\boldsymbol{\omega}(\hat\omega,\psi)$ be the industry of the firm at node ψ in the supply tree for a firm in industry $\hat\omega$. Abusing notation, we also let $\boldsymbol{\omega}(j)$ denote j's industry.

For a supply tree for a firm j in industry ω, there is a path from the root (firm j) to the node at ψ. Each edge along the path from the root to the node has a buyer b and a supplier s, and in particular an output elasticity $\alpha_{\omega(b),\omega(s)}$ of the buyer's production function with repect to the supplier's input. Let $\sigma_{\omega\psi}$ be the product of those output elasticities along the path. For completeness, we say that if ψ is the root of the tree then the product has no terms and we define $\sigma_{\omega\psi} = 1$.

Note that if since each production function has constant returns to scale, the sum of the output-elasticities of labor across all nodes in a supply tree is one. That is, for firm j in industry ω,

$$\sum_{\psi \in \Psi_\omega} \sigma_{\omega\psi} \alpha_{\omega(j,\psi),l} = 1$$

For firm j in industry ω, let $z(j,\psi)$ be the match-specific productivity associated with using the supplier at node ψ in the supply tree Ψ_ω. We define $q_{j\hat{\omega}}$ to be the effective productivity the of the supply tree that produces input $\hat{\omega}$ for the firm. If firm j uses supplier s for input $\hat{\omega}$, then

$$q_{j\hat{\omega}} = \prod_{\psi \in \Psi_{\hat{\omega}}} z(s,\psi)^{\sigma_{\hat{\omega}\psi}}$$

Note that this implies the iterative definition (6) in the main text, that if j uses supplier s for input $\hat{\omega}$, then

$$q_{j\hat{\omega}} = z_{js} \prod_{\tilde{\omega}} q_{s\tilde{\omega}}^{\alpha_{\hat{\omega}\tilde{\omega}}}$$

We will now build toward defining a feasible allocation. Any allocation has a supply tree representation. A supply tree representation is a decomposition of production of each firm into the output used for each retailer's supply tree that the firm is in. Formally, for each retailer r and node in the retailer's supply tree $\psi \in \Psi_{\omega(r)}$, let $y(r,\psi)$ be the output used by the firm at node ψ in the eventual production of the retailer's good, while $l(r,\psi)$ and $\{x_{\hat{\omega}}(r,\psi)\}_{\hat{\omega}}$ are the labor and intermediate inputs used by that firm toward the production of the retailer's good. The supply tree representation of the allocation is $\{y(r,\psi), l(r,\psi), \{x_{\hat{\omega}}(r,\psi)\}_{\hat{\omega}}\}_{r\in R,\psi\in\Psi_{\omega(r)}}$. We will also sometimes use the alternative notation; if firm j is at node ψ in the supply tree for retailer r, then

$$\begin{aligned} y_j(r) &\equiv y(r,\psi) \\ l_j(r) &\equiv l(r,\psi) \\ x_{j,\hat{\omega}}(r) &\equiv x_{\hat{\omega}}(r,\psi) \end{aligned}$$

The supply tree must satisfy several constraints. First, for any buyer supplier pair b and s where the supplier is in industry $\hat{\omega}$,

$$x_{b,\hat{\omega}}(r) = y_s(r)$$

Second, for any firm j, the total inputs and output across supply trees cannot must equal the firms total inputs and outputs.

$$\begin{aligned} l_j &= \sum_{r\in\mathcal{R}_j} l_j(r) \\ y_j &= \sum_{r\in\mathcal{R}_j} y_j(r) \\ x_{j,\mathfrak{s}(j,\hat{\omega})} &= \sum_{r\in\mathcal{R}_j} x_{b,\hat{\omega}}(r) \end{aligned}$$

Finally, outputs and inputs at each node must be consistent with the production function

$$y_j(r) = A_\omega l_j(r)^{\alpha_{\omega l}} \prod_{\hat{\omega}} \left[z_{j,\mathfrak{s}(j,\hat{\omega})} x_{j,\hat{\omega}}(r)\right]^{\alpha_{\omega\hat{\omega}}}$$

For an allocation to be feasible, it must be that for any node ψ in the supply chain

$$A_\omega l_j(r)^{\alpha_{\omega l}} \prod_{\tilde{\psi}\in\Psi_{\omega(r)}|\tilde{\psi}>\psi} \left[z\left(r,\tilde{\psi}\right) A_{\omega\left(r,\tilde{\psi}\right)} l\left(r,\tilde{\psi}\right)^{\alpha_{\omega\left(r,\tilde{\psi}\right),l}}\right]^{\sigma_{\psi,\tilde{\psi}}} \geq y_j\left(r\right)$$

To be feasible, it must be that for any node $\psi \in \Psi_{\omega(r)}$ in the

$$A_{\omega(r,\psi)} l(r,\psi)^{\alpha_{\omega(r,\psi),l}} \prod_{\tilde{\psi}\in\Psi_{\omega(r)}|\tilde{\psi}>\psi} \left[z\left(r,\tilde{\psi}\right) A_{\omega\left(r,\tilde{\psi}\right)} l\left(r,\tilde{\psi}\right)^{\alpha_{\omega\left(r,\tilde{\psi}\right),l}}\right]^{\sigma_{\psi,\tilde{\psi}}} \leq y\left(r,\psi\right)$$

Lemma 1 *Consider a pairwise stable arrangement and the associated supply tree representation. For any firm j that is in node ψ of the supply tree of retailer r, and any node $\tilde{\psi}$ that is weakly further upstream, it must be that*

$$wl\left(r,\tilde{\psi}\right) = \alpha_{\omega\left(r,\tilde{\psi}\right),l}\sigma_{\psi\tilde{\psi}} c_j y_j(r)$$

Proof. Given an arrangement, we can reformulate firm j's in industry ω problem into choosing how to allocate inputs for production for each supply tree that uses it. If the arrangement dictates that firm j uses inputs $\left\{x_{j,\mathfrak{s}(j,\hat{\omega})}\right\}_{\hat{\omega}}$, then we can express j's expenditure minimization problem as choosing a quantity of labor and of each input in order to make output for each supply tree:

$$\min_{\left\{l_j(r),x_{j,\hat{\omega}}(r)\right\}_{r\in\mathcal{R}_j}} \sum_{r\in\mathcal{R}_j} wl_j\left(r\right)$$

subject to $\sum_{r\in\mathcal{R}_j} x_{j,\hat{\omega}}(r) \leq x_{j,\mathfrak{s}(j,\hat{\omega})}$ for each input $\hat{\omega}$ and $y_j(r) \leq A_\omega l_j(r)^{\alpha_{\omega l}} \prod_{\hat{\omega}} \left[z_{j,\mathfrak{s}(j,\hat{\omega})} x_{j,\hat{\omega}}(r)\right]^{\alpha_{\omega\hat{\omega}}}$ for each retailer $r \in \mathcal{R}_j$. Letting $\eta_j(r)$ be the multiplier on the production constraint for each buyer and $\lambda_{j\hat{\omega}}$ be the multiplier on the input supply constraint from each supplier, the result of this cost minimization problem is

$$
\begin{aligned}
wl_j(r) &= \alpha_{\omega l}\eta_j(r)y_j(r), \;\; \forall r \in \mathcal{R}_j \\
\lambda_{j\hat{\omega}}x_{j,\hat{\omega}}(r) &= \alpha_{\omega\hat{\omega}}\eta_j(r)y_j(r), \;\; \forall \hat{\omega}, r \in \mathcal{R}_j
\end{aligned}
$$

Summing each of these across retailers, using $wl_j = \alpha_{\omega l}c_j y_j$ from Claim 1, and using $c_s x_{js} = \alpha_{\omega\hat{\omega}}y_j c_j$ from Claim 2 give and $\eta_j(r) = c_j$ and $\lambda_{j\hat{\omega}} = c_{\mathfrak{s}(j,\hat{\omega})}$. As a result, we have

$$
\begin{aligned}
wl_j(r) &= \alpha_{\omega l}c_j y_j(r), \;\; \forall r \in \mathcal{R}_j \\
c_{\mathfrak{s}(j,\hat{\omega})}x_{j,\hat{\omega}}(r) &= \alpha_{\omega\hat{\omega}}c_j y_j(r), \;\; \forall \hat{\omega}, r \in \mathcal{R}_j
\end{aligned}
$$

For any buyer supplier pair, b and s in respective industries ω and $\hat{\omega}$, $x_{b,\hat{\omega}}(r) = y_s(r)$. Combining these and (if necessary) iterating the latter two forward gives

$$
wl\left(r, \tilde{\psi}\right) = \alpha_{\mathrm{\omega}\left(r,\tilde{\psi}\right),l}\sigma_{\psi\tilde{\psi}}c_j y_j(r)
$$

■

Lemma 2 *For any industry ω,*

$$
A_\omega \alpha_{\omega l}^{\alpha_{\omega l}} \prod_{\hat{\omega}} \left(\prod_{\psi \in \Psi_{\hat{\omega}}} \left[A_{\mathrm{\omega}(\hat{\omega},\psi)} \left[\alpha_{\omega\hat{\omega}}\sigma_{\hat{\omega}\psi}\alpha_{\mathrm{\omega}(\hat{\omega},\psi),l} \right]^{\alpha_{\mathrm{\omega}(\hat{\omega},\psi),l}} \right]^{\sigma_{\hat{\omega}\psi}} \right)^{\alpha_{\omega\hat{\omega}}} = 1
$$

Proof. First, using the definition of A_ω, we can simplify the left hand side of the expression to get

$$
A_\omega \alpha_{\omega l}^{\alpha_{\omega l}} \prod_{\hat{\omega}} \left(\prod_{\psi \in \Psi_{\hat{\omega}}} \left[A_{\mathrm{\omega}(\hat{\omega},\psi)} \left[\alpha_{\omega\hat{\omega}}\sigma_{\hat{\omega}\psi}\alpha_{\mathrm{\omega}(\hat{\omega},\psi),l} \right]^{\alpha_{\mathrm{\omega}(\hat{\omega},\psi),l}} \right]^{\sigma_{\hat{\omega}\psi}} \right)^{\alpha_{\omega\hat{\omega}}} = \prod_{\hat{\omega}} \left(\prod_{\psi \in \Psi_{\hat{\omega}}} A_{\mathrm{\omega}(\hat{\omega},\psi)}^{\sigma_{\hat{\omega}\psi}} \left[\alpha_{\mathrm{\omega}(\hat{\omega},\psi),l}\sigma_{\hat{\omega}\psi} \right]^{\alpha_{\mathrm{\omega}(\hat{\omega},\psi),l}\sigma_{\hat{\omega}\psi}} \right)^{\alpha_{\omega\hat{\omega}}}
$$

Next, we will show that each term in the weighted product is 1. That is, for each input $\hat{\omega}$, we will show that $\prod_{\psi \in \Psi_{\hat{\omega}}} A_{\mathrm{\omega}(\hat{\omega},\psi)}^{-\sigma_{\hat{\omega}\psi}} = \prod_{\psi \in \Psi_{\hat{\omega}}} \left[\alpha_{\mathrm{\omega}(\hat{\omega},\psi),l}\sigma_{\hat{\omega}\psi} \right]^{\alpha_{\mathrm{\omega}(\hat{\omega},\psi),l}\sigma_{\hat{\omega}\psi}}$. To get this, we have from the definition of $A_{\mathrm{\omega}(\hat{\omega},\psi)}$,

$$
\begin{aligned}
\prod_{\psi \in \Psi_{\hat{\omega}}} A_{\mathrm{\omega}(\hat{\omega},\psi)}^{-\sigma_{\hat{\omega}\psi}} &= \prod_{\psi \in \Psi_{\hat{\omega}}} \left(\alpha_{\mathrm{\omega}(\hat{\omega},\psi),l}^{\alpha_{\mathrm{\omega}(\hat{\omega},\psi),l}} \prod_{\tilde{\omega}} \alpha_{\mathrm{\omega}(\hat{\omega},\psi),\tilde{\omega}}^{\alpha_{\mathrm{\omega}(\hat{\omega},\psi),\tilde{\omega}}} \right)^{\sigma_{\hat{\omega}\psi}} \\
&= \left(\prod_{\psi \in \Psi_{\hat{\omega}}} \alpha_{\mathrm{\omega}(\hat{\omega},\psi),l}^{\alpha_{\mathrm{\omega}(\hat{\omega},\psi),l}\sigma_{\hat{\omega}\psi}} \right) \left(\prod_{\psi \in \Psi_{\hat{\omega}}} \prod_{\tilde{\omega}} \alpha_{\mathrm{\omega}(\hat{\omega},\psi),\tilde{\omega}}^{\alpha_{\mathrm{\omega}(\hat{\omega},\psi),\tilde{\omega}}\sigma_{\hat{\omega}\psi}} \right)
\end{aligned}
$$

Using $\sigma_{\hat{\omega}\psi}\alpha_{\omega(\hat{\omega},\psi),\tilde{\omega}} = \sum_{\tilde{\psi}\in\Psi_{\tilde{\omega}}} \sigma_{\hat{\omega}\tilde{\psi}}\alpha_{\omega\left(\hat{\omega},\tilde{\psi}\right),l}$, the second term can be expressed as

$$\begin{aligned}
\prod_{\psi\in\Psi_{\hat{\omega}}}\prod_{\tilde{\omega}}\alpha_{\omega(\hat{\omega},\psi),\tilde{\omega}}^{\alpha_{\omega(\hat{\omega},\psi),\tilde{\omega}}\sigma_{\hat{\omega}\psi}} &= \prod_{\psi\in\Psi_{\hat{\omega}}}\prod_{\tilde{\omega}}\alpha_{\omega(\hat{\omega},\psi),\tilde{\omega}}^{\sum_{\tilde{\psi}\in\Psi_{\tilde{\omega}}}\sigma_{\hat{\omega}\tilde{\psi}}\alpha_{\omega(\hat{\omega},\tilde{\psi}),l}} \\
&= \prod_{\psi\in\Psi_{\hat{\omega}}}\prod_{\tilde{\omega}}\prod_{\tilde{\psi}\in\Psi_{\tilde{\omega}}}\alpha_{\omega(\hat{\omega},\psi),\tilde{\omega}}^{\sigma_{\hat{\omega}\tilde{\psi}}\alpha_{\omega(\hat{\omega},\tilde{\psi}),l}} \\
&= \prod_{\tilde{\psi}\in\Psi_{\omega}}\sigma_{\hat{\omega}\tilde{\psi}}^{\sigma_{\hat{\omega}\tilde{\psi}}\alpha_{\omega(\hat{\omega},\tilde{\psi}),l}}
\end{aligned}$$

where the last line follows from switching the order of the products and the definition of $\sigma_{\hat{\omega}\tilde{\psi}}$. Together, these yield

$$\prod_{\psi\in\Psi_{\hat{\omega}}} A_{\omega(\hat{\omega},\psi)}^{-\sigma_{\hat{\omega}\psi}} = \prod_{\psi\in\Psi_{\hat{\omega}}}\left[\alpha_{\omega(\hat{\omega},\psi),l}\sigma_{\hat{\omega}\psi}\right]^{\alpha_{\omega(\hat{\omega},\psi),l}\sigma_{\hat{\omega}\psi}}$$

■

Proposition 7 *Consider a feasible, pairwise stable arrangement. For any firm j in industry ω with $y_j > 0$,*

$$\frac{w}{c_j} \leq \prod_{\hat{\omega}} q_{j\hat{\omega}}^{\alpha_{\omega\hat{\omega}}}$$

Proof. If $c_j = \infty$, then the conclusion is immediate. Otherwise, if the arrangement is feasible, then there is a retailer r with supply tree $\Psi_{\omega(r)}$ such that $y_r > 0$ and such that j is at node $\psi \in \Psi_{\omega(r)}$. Feasibility implies

$$A_\omega l(r,\psi)^{\alpha_{\omega l}} \prod_{\tilde{\psi}\in\Psi_{\omega(r)}|\tilde{\psi}>\psi}\left[z\left(r,\tilde{\psi}\right)A_{\omega\left(r,\tilde{\psi}\right)}l\left(r,\tilde{\psi}\right)^{\alpha_{\omega\left(r,\tilde{\psi}\right),l}}\right]^{\sigma_{\psi,\tilde{\psi}}} \geq y_j(r)$$

Pairwise stability implies that for any node $\tilde{\psi} \geq \psi$

$$l\left(r,\tilde{\psi}\right) = \frac{\alpha_{\omega\left(r,\tilde{\psi}\right),l}\sigma_{\psi\tilde{\psi}}c_j y_j(r)}{w}$$

Eliminating $l\left(r,\tilde{\psi}\right)$ from these equations yields

$$A_\omega\left[\frac{\alpha_{\omega l}c_j y_j(r)}{w}\right]^{\alpha_{\omega l}} \prod_{\tilde{\psi}\in\Psi_{\omega(r)}|\tilde{\psi}>\psi}\left[z\left(r,\tilde{\psi}\right)A_{\omega\left(r,\tilde{\psi}\right)}\left[\frac{\alpha_{\omega\left(r,\tilde{\psi}\right),l}\sigma_{\psi\tilde{\psi}}c_j y_j(r)}{w}\right]^{\alpha_{\omega\left(r,\tilde{\psi}\right),l}}\right]^{\sigma_{\psi,\tilde{\psi}}} \geq y_j(r)$$

Multiplying both sides by $\frac{w}{c_j y_j(r)}$ and using $\alpha_{\omega l} + \sum_{\tilde{\psi}\in\Psi_{\omega(r)}|\tilde{\psi}>\psi}\sigma_{\psi,\tilde{\psi}}\alpha_{\omega\left(r,\tilde{\psi}\right),l} = 1$ gives

$$A_\omega\alpha_{\omega l}^{\alpha_{\omega l}} \prod_{\tilde{\psi}\in\Psi_{\omega(r)}|\tilde{\psi}>\psi}\left[z\left(r,\tilde{\psi}\right)A_{\omega\left(r,\tilde{\psi}\right)}\left[\alpha_{\omega\left(r,\tilde{\psi}\right),l}\sigma_{\psi\tilde{\psi}}\right]^{\alpha_{\omega\left(r,\tilde{\psi}\right),l}}\right]^{\sigma_{\psi,\tilde{\psi}}} \geq \frac{w}{c_j}$$

Next, note that for any r downstream of j,

$$\prod_{\tilde{\psi}\in\Psi_{\omega(r)}|\tilde{\psi}>\psi} z\left(r,\tilde{\psi}\right)^{\sigma_{\psi,\tilde{\psi}}} = \prod_{\hat{\omega}} q_{j\hat{\omega}}^{\alpha_{\omega\hat{\omega}}}$$

Lastly, we can rearrange the remaining terms on the right hand side and use lemma 2 to get

$$\begin{aligned}
& A_\omega \alpha_{\omega l}^{\alpha_{\omega l}} \prod_{\tilde{\psi}\in\Psi_{\omega(r)}|\tilde{\psi}>\psi} \left[A_{\omega\left(r,\tilde{\psi}\right)} \left[\alpha_{\omega\left(r,\tilde{\psi}\right),l}\sigma_{\psi\tilde{\psi}}\right]^{\alpha_{\omega\left(r,\tilde{\psi}\right),l}}\right]^{\sigma_{\psi,\tilde{\psi}}} \\
= \; & A_\omega \alpha_{\omega l}^{\alpha_{\omega l}} \prod_{\hat{\omega}} \left(\prod_{\tilde{\psi}\in\Psi_{\hat{\omega}}} \left[A_{\omega\left(\hat{\omega},\tilde{\psi}\right)} \left[\alpha_{\omega\hat{\omega}}\alpha_{\omega\left(\hat{\omega},\tilde{\psi}\right),l}\sigma_{\hat{\omega}\tilde{\psi}}\right]^{\alpha_{\tilde{\omega}\left(\hat{\omega},\tilde{\psi}\right),l}}\right]^{\sigma_{\hat{\omega}\tilde{\psi}}}\right)^{\alpha_{\omega\hat{\omega}}} \\
= \; & 1
\end{aligned}$$

Together, these last three expressions deliver $\prod_{\hat{\omega}} q_{j\hat{\omega}}^{\alpha_{\omega\hat{\omega}}} \geq \frac{w}{c_j}$. ■

Claim 5 *Let $\bar{l}(r) = \sum_{\psi\in\Psi_R} l(\psi, r)$ be total labor used across the supply for r. For any pairwise stable arrangement, $w\bar{l}(r) = c_r y_r$*

Proof. Lemma 1 gives for the retailers output gives

$$wl\left(r,\tilde{\psi}\right) = \alpha_{\omega\left(r,\tilde{\psi}\right),l}\sigma_{R\tilde{\psi}} c_r y_r$$

Summing over all nodes gives

$$w\bar{l}(r) = \sum_{\psi\in\Psi_R} \alpha_{\omega\left(r,\tilde{\psi}\right),l}\sigma_{R\tilde{\psi}} c_r y_r$$

The conclusion follows from the fact that $\sum_{\psi\in\Psi_R} \alpha_{\omega\left(r,\tilde{\psi}\right),l}\sigma_{R\tilde{\psi}} = 1$. ■

Claim 6 *For any pairwise stable arrangement,*

$$Y = (1-\eta) L \left(\int_{r\in\mathcal{R}} \left(\frac{c_r}{w}\right)^{1-\varepsilon} dr\right)^{\frac{1}{\varepsilon-1}}$$

Proof. Starting with $\bar{l}(r) = \frac{c_r}{w} y_r$, total labor used in production is

$$(1-\eta) L = \int_{r\in\mathcal{R}} \bar{l}(r)\, dr = \int_{r\in\mathcal{R}} \frac{c_r y_r}{w} dr$$

Using the optimal price $p_r = \frac{\varepsilon}{\varepsilon-1} c_r$ and demand $y_r = Y\left(\frac{p_r}{P}\right)^{-\varepsilon}$, we have $c_r y_r = \frac{\varepsilon-1}{\varepsilon} p_r y_r = \frac{\varepsilon-1}{\varepsilon} p_r Y \left(\frac{p_r}{P}\right)^{-\varepsilon} = \frac{\varepsilon-1}{\varepsilon} PY \left(\frac{p_r}{P}\right)^{1-\varepsilon}$. Using this along with $P = \left(\int_{r\in\mathcal{R}} p_r^{1-\varepsilon} dr\right)^{\frac{1}{1-\varepsilon}}$ gives

$$(1-\eta) L = \int_{r\in\mathcal{R}} \frac{\frac{\varepsilon-1}{\varepsilon} PY \left(\frac{p_r}{P}\right)^{1-\varepsilon}}{w} dr = \frac{\frac{\varepsilon-1}{\varepsilon} PY}{w}$$

Noting that $\frac{\varepsilon-1}{\varepsilon}\frac{P}{w} = \frac{\varepsilon-1}{\varepsilon}\frac{\left(\int_{r\in\mathcal{R}} p_r^{1-\varepsilon} dr\right)^{\frac{1}{1-\varepsilon}}}{w} = \left(\int_{r\in\mathcal{R}} \left(\frac{c_r}{w}\right)^{1-\varepsilon} dr\right)^{\frac{1}{1-\varepsilon}}$, solving for Y gives

$$Y = (1-\eta) L \left(\int_{r\in\mathcal{R}} \left(\frac{c_r}{w}\right)^{1-\varepsilon} dr\right)^{\frac{1}{\varepsilon-1}}$$

■

D.1.3 Countable Stability

Lemma 3 *In any countably stable equilibrium, then for any firm j such that $y_j > 0$,*

$$c_j = w \prod_{\hat{\omega}} q_{j\hat{\omega}}^{-\alpha_{\omega\hat{\omega}}}$$

Proof. Suppose not. $c_j > w \prod_{\hat{\omega}} q_{j\hat{\omega}}^{-\alpha_{\omega\hat{\omega}}}$, then there is a dominating deviation by a coalition comprised of all of the firms in j's supply tree that can produce the same output using less total labor, so that with transfers, all firms in the coalition would be better off. ■

Lemma 4 *In any countably stable equilibrium, $\tau_{bs} \geq 0$.*

Proof. Suppose that there is a buyer supplier pair in which $\tau_{bs} < 0$. Then the coalition of the supply tree for firm s has a dominating deviation in which s drops the buyer, and all firms in the coalition scaling back production just enough so that all other obligations to other buyers are met. The cost saving to the coalition would be $c_s x_{bs}$, which is larger than T_{bs}. ■

Lemma 5 *If $\pi_j \geq 0$ for all j, $\tau_{bs} \geq 0$ for all pairs, and $c_j = w \prod_{\hat{\omega}} q_{j\hat{\omega}}^{-\alpha_{\omega\hat{\omega}}}$ for all j, then no countable coalition has a dominating deviation.*

Proof. Consider any coalition. The coalition cannot reduce its total labor cost or increase any retailer's profit from the household by shifting quantities. If any supplier outside the coalition is dropped, any supply tree that relied on that supplier would have to reduce output to zero, so the deviation is either eliminate profit or is infeasible. ■

D.1.4 Bargaining Weights

We now characterize a set of equilibria indexed by $\Upsilon \in [0,1]$, which can be interpreted as the supplier's bargaining power.

Claim 7 *For any $\Upsilon \in [0,1]$, there is a countably stable equilibrium in which*

$$\tau_{j,\mathfrak{s}(j,\hat{\omega})} = \Upsilon \alpha_{\omega\hat{\omega}} \left(\sum_{b\in\mathcal{B}_j} \tau_{bj}\right)$$

if j is a producer firm or

$$\tau_{j,\mathfrak{s}(j,\hat{\omega})} = \Upsilon \alpha_{\omega\hat{\omega}} \left(p_j - c_j\right) y_j$$

if j is a retailer

Proof. In this case, $\tau_{bs} \geq 0$ for every buyer-supplier pair and each firm's profit is weakly positive, $\pi_j \geq 0$. ■

Claim 8 *If $\Upsilon = 1$, each firm in industry ω has a ratio of revenue to cost of $\frac{\varepsilon}{\varepsilon - \alpha_{\omega l}}$ and a ratio of revenue to labor of $\frac{w}{\alpha_{\omega l}} \frac{\varepsilon}{\varepsilon - 1}$*

Proof. Retailer j's revenue is $p_j y_j$. Since $\tau_{j,\mathfrak{s}(j,\hat{\omega})} = \alpha_{\omega\hat{\omega}} \left(\left(p_j - c_j\right) y_j\right)$ for each of its suppliers, its total cost is

$$\begin{aligned} wl_j + \sum_{\hat{\omega}} T_{j,\mathfrak{s}(j,\hat{\omega})} &= wl_j + \sum_{\hat{\omega}} \left[c_{\mathfrak{s}(j,\hat{\omega})} x_{j,\mathfrak{s}(j,\hat{\omega})} + \tau_{j,\mathfrak{s}(j,\hat{\omega})}\right] \\ &= \alpha_{\omega l} c_j y_j + \sum_{\hat{\omega}} \left[\alpha_{\omega\hat{\omega}} c_j y_j + \alpha_{\omega\hat{\omega}} \left(p_j - c_j\right) y_j\right] \\ &= c_j y_j + \left(1 - \alpha_{\omega l}\right) \left(p_j - c_j\right) y_j \end{aligned}$$

The retailer's ratio of revenue to cost is then

$$\begin{aligned} \frac{p_j y_j}{c_j y_j + \left(1 - \alpha_{\omega l}\right) \left(p_j - c_j\right) y_j} &= \frac{\frac{\varepsilon}{\varepsilon - 1} c_j y_j}{c_j y_j + \left(1 - \alpha_{\omega l}\right) \left(\frac{\varepsilon}{\varepsilon - 1} c_j - c_j\right) y_j} \\ &= \frac{\frac{\varepsilon}{\varepsilon - 1}}{1 + \left(1 - \alpha_{\omega l}\right) \left(\frac{\varepsilon}{\varepsilon - 1} - 1\right)} \\ &= \frac{\varepsilon}{\left(1 - \alpha_{\omega l}\right) \varepsilon + \left(\varepsilon - 1\right) \alpha_{\omega l}} \\ &= \frac{\varepsilon}{\varepsilon - \alpha_{\omega l}} \end{aligned}$$

To characterize transfers among producer firms we turn again to the supply tree representation. For node ψ in the supply tree for j let $\tau^b\left(j, \psi\right)$ be the transfor of surplus for the supply tree from the firm at that node from its buyer. Similarly, let $\tau_{\hat{\omega}}\left(j, \psi\right)$ be the transfer of surplus from the firm at that node to its supplier. With $\Upsilon = 1$, this must satisfy τ

$$\tau_{\hat{\omega}}\left(j, \psi\right) = \alpha_{\omega(j,\psi),\hat{\omega}} \tau^b\left(j, \psi\right)$$

We now proceed by induction to show that for each producing firm, the ratio of revenue to $\frac{\tau_{j,b}(r)}{c_j y_j(r)} = \frac{1}{\varepsilon - 1}$. If firm j in industry $\hat{\omega}$ supplies the retailer directly, then the payment of surplus to the supplier is

$$\tau_{j,b}\left(r\right) = \alpha_{R\hat{\omega}} y_r \left(p_r - c_r\right) = \frac{1}{\varepsilon - 1} \alpha_{\omega(j,\psi),\hat{\omega}} y_r c_r$$

Pairwise stability implies. $c_j y_j(r) = \alpha_{R\hat{\omega}} c_r y_r$. Together, these imply that

$$\frac{\tau_{j,b}(r)}{c_j y_j(r)} = \frac{\frac{1}{\varepsilon-1}\alpha_{R\hat{\omega}} y_r c_r}{\alpha_{R\hat{\omega}} c_r y_r} = \frac{1}{\varepsilon - 1}$$

Now consider firm any producer firm j in industry ω that uses supplier s for input $\hat{\omega}$. With $\Upsilon = 1$, the payment of surplus from j to s for the supply tree for r is $\tau_{s,b}(r) = \alpha_{\omega\hat{\omega}} \tau_{j,b}(r)$. Pairwise stability implies $c_j y_j(r) = \alpha_{\omega\hat{\omega}} c_j y_j(r)$. Together these yield

$$\frac{\tau_{s,b}(r)}{c_s y_s(r)} = \frac{\alpha_{\omega\hat{\omega}} \tau_{j,b}(r)}{\alpha_{\omega\hat{\omega}} c_j y_j(r)} = \frac{\tau_{j,b}(r)}{c_j y_j(r)} = \frac{1}{\varepsilon - 1}$$

Now, for producer firm j in industry ω is

$$\begin{aligned}
\text{Revenue}_j &= \sum_{r \in \mathcal{R}_j} \left[\tau_{j,b}(r) + c_j x_j(r)\right] = \sum_{r \in \mathcal{R}_j} \left[\frac{1}{\varepsilon - 1} c_j x_j(r) + c_j x_j(r)\right] = \frac{\varepsilon}{\varepsilon - 1} c_j y_j \\
l_j &= \frac{\alpha_{\omega l} c_j y_j}{w} \\
\text{Expenditure}_j &= w l_j + \sum_{r \in \mathcal{R}_j} \sum_{\hat{\omega}} \left[\tau_{\mathfrak{s}(j,\hat{\omega}),b}(r) + c_{\mathfrak{s}(j,\hat{\omega})} x_{\mathfrak{s}(j,\hat{\omega})}(r)\right] \\
&= w l_j + \sum_{r \in \mathcal{R}_j} \sum_{\hat{\omega}} \left[\frac{1}{\varepsilon - 1} c_{\mathfrak{s}(j,\hat{\omega})} x_{\mathfrak{s}(j,\hat{\omega})}(r) + c_{\mathfrak{s}(j,\hat{\omega})} x_{\mathfrak{s}(j,\hat{\omega})}(r)\right] \\
&= \alpha_{\omega l} c_j y_j + \sum_{\hat{\omega}} \frac{\varepsilon}{\varepsilon - 1} \alpha_{\omega\hat{\omega}} c_j y_j \\
&= \left(\alpha_{\omega l} + (1 - \alpha_{\omega l}) \frac{\varepsilon}{\varepsilon - 1}\right) c_j y_j
\end{aligned}$$

As a result, the ratio of revenue to cost is

$$\frac{\text{Revenue}_j}{\text{Expenditure}_j} = \frac{\frac{\varepsilon}{\varepsilon-1} c_j y_j}{\left(\alpha_{\omega l} + (1 - \alpha_{\omega l}) \frac{\varepsilon}{\varepsilon-1}\right) c_j y_j} = \frac{\varepsilon}{\varepsilon - \alpha_{\omega l}}$$

while the ratio of revenue to labor is

$$\frac{\text{Revenue}_j}{l_j} = \frac{\frac{\varepsilon}{\varepsilon-1} c_j y_j}{\frac{\alpha_{\omega l} c_j y_j}{w}} = \frac{w}{\alpha_{\omega l}} \frac{\varepsilon}{\varepsilon - 1}$$

■

D.2 Dynamics in the Simple Model

First, we define the infinitesimal generator of the process. Let $m(x) \equiv M_t(x, 0)$, where M_t is the partial derivative of M with respect to its second argument.

Lemma 6

$$m(x) = -\phi \sum_{k=1}^{\infty} x^{-\beta\alpha^{-k}}$$

Proof. (Heuristic)

For a short enough time period t, there are two ways that a firm's cost can fall by a factor larger tan x: either the firm finds a new supplier that delivers a jump in effective cost larger than $x^{1/\alpha}$, or the existing supplier's efficiency improves by more than $x^{1/\alpha}$ (the full proof shows that the probability of some mixture of the two events is negligible).

$$\begin{aligned} M(x,t) &= \left\{ M\left(x^{1/\alpha}, t\right)\right\} \left\{ e^{-t\phi \int_{x^{1/\alpha}}^{\infty} \beta b^{-\beta-1} db} \right\} \\ &= M\left(x^{1/\alpha}, t\right) e^{-t\phi x^{-\beta/\alpha}} \end{aligned}$$

Differentiating and evaluating at $t = 0$

$$\begin{aligned} M_t(x,0) &= \frac{d}{dt}\left\{ M\left(x^{1/\alpha}, t\right) e^{-t\phi x^{-\beta/\alpha}} \right\}\Big|_{t=0} \\ &= M_t\left(x^{1/\alpha}, t\right) e^{-t\phi x^{-\beta/\alpha}} - \phi x^{-\beta/\alpha} M\left(x^{1/\alpha}, t\right) e^{-t\phi x^{-\beta/\alpha}}\Big|_{t=0} \\ &= M_t\left(x^{1/\alpha}, 0\right) - \phi x^{-\beta/\alpha} \end{aligned}$$

where the last line used $M\left(x^{1/\alpha}, 0\right) = 1$.

Next, using the fact that $\lim_{x\to\infty} M_t(x,0) = 0$, we can compute M recursively:

$$\begin{aligned} m(x) &= -\phi x^{-\beta/\alpha} + m\left(x^{1/\alpha}\right) \\ &= -\phi x^{-\beta/\alpha} - \phi x^{-\beta/\alpha^2} + m\left(x^{1/\alpha^2}\right) \\ &= -\sum_{k=1}^{K} \phi x^{-\beta\alpha^{-k}} + m\left(x^{\alpha^{-K}}\right) \\ &\to -\phi \sum_{k=1}^{\infty} x^{-\beta\alpha^{-k}} \end{aligned}$$

■

Proof. (Full)

What is the probability that a firm's efficiency increase by weakly less than the proportion x in an interval of length t? This could happen either the firm does not find a new supplier and the existing supplier improves by no more than $x^{1/\alpha}$, an even which occurs with probability $e^{-\phi t} M\left(x^{1/\alpha}, t\right)$, or if there is a jump with increment z at some time $\tau \in [0,t]$, which occurs with density $\beta z^{-\beta-1}\phi e^{-\phi\tau}$, that the new supplier improved by y between τ and t, and that the initial existing supplier improved by less than $\frac{x^{1/\alpha}}{yz}$ between 0 and τ. Thus

$$M(x,t) = e^{-\phi t} M\left(x^{1/\alpha}, t\right) + \int_0^t \phi e^{-\phi\tau} \int_1^{x^{1/\alpha}} \int_1^{\frac{x^{1/\alpha}}{y}} M\left(\frac{x^{1/\alpha}}{yz}, \tau\right) \beta z^{-\beta-1} M_x\left(y, t-\tau\right) dz dy d\tau$$

Taking the derivative with respect to t gives

$$\begin{aligned} M_t(x,t) &= -\phi e^{-\phi t} M\left(x^{1/\alpha}, t\right) + e^{-\phi t} M_t\left(x^{1/\alpha}, t\right) \\ &\quad + \phi e^{-\phi t} \int_1^{x^{1/\alpha}} \int_1^{\frac{x^{1/\alpha}}{y}} M\left(\frac{x^{1/\alpha}}{yz}, t\right) \beta z^{-\beta-1} M_x(y,0) dz dy \\ &\quad + \int_0^t \phi e^{-\phi\tau} \int_1^{x^{1/\alpha}} \int_1^{\frac{x^{1/\alpha}}{y}} M\left(\frac{x^{1/\alpha}}{yz}, \tau\right) \beta z^{-\beta-1} M_{xt}(y, t-\tau) dz dy d\tau \end{aligned}$$

Evaluating this at $t = 0$ and gives

$$M_t(x,0) = -\phi M\left(x^{1/\alpha}, 0\right) + M_t\left(x^{1/\alpha}, 0\right) + \phi \int_1^{x^{1/\alpha}} \int_1^{\frac{x^{1/\alpha}}{y}} M\left(\frac{x^{1/\alpha}}{yz}, 0\right) \beta z^{-\beta-1} M_x\left(y, 0\right) dz dy$$

Using the fact that $M\left(x, 0\right) = 1$, $\forall x \geq 1$ and that $M_x\left(y, 0\right)$ is the Dirac delta function at $y = 1$, this is

$$\begin{aligned} M_t(x,0) &= -\phi + M_t\left(x^{1/\alpha}, 0\right) + \phi \int_1^{x^{1/\alpha}} \left[1 - \left(\frac{x^{1/\alpha}}{y}\right)^{-\beta}\right] M_x\left(y, 0\right) dy \\ &= -\phi + M_t\left(x^{1/\alpha}, 0\right) + \phi\left(1 - x^{-\beta/\alpha}\right) \\ &= M_t\left(x^{1/\alpha}, 0\right) - \phi x^{-\beta/\alpha} \end{aligned}$$

Next, using the fact that $\lim_{x\to\infty} M_t(x,0) = 0$, we can compute M recursively:

$$\begin{aligned} m(x) &= -\phi x^{-\beta/\alpha} + m\left(x^{1/\alpha}\right) \\ &= -\phi x^{-\beta/\alpha} - \phi x^{-\frac{\beta}{\alpha^2}} + m\left(x^{\frac{1}{\alpha^2}}\right) \\ &= -\sum_{k=1}^{K} \phi x^{-\beta\alpha^{-k}} + m\left(x^{\alpha^{-K}}\right) \\ &\to -\phi \sum_{k=1}^{\infty} x^{-\beta\alpha^{-k}} \end{aligned}$$

■

To get at the distribution of efficiency growth we define $\varphi\left(s,t\right)$ to be the Mellin transform of $M\left(x,t\right)$, i.e., $\varphi\left(s,t\right) \equiv \int_1^\infty x^{-s} M_x\left(x,t\right) dx$.

Claim 9 $\varphi(s,t) = e^{-\phi t \sum_{k=1}^{\infty} \frac{s}{\beta\alpha^{-k}+s}}$

Proof. We first derive an expression for the time derivative of the Mellin transform at $t = 0$. To do this, we first integrate by parts

$$\begin{aligned}\varphi(s,t) &= \int_1^\infty x^{-s} M_x(x,t)\,dx = M(x,t)\,x^{-s}\big|_1^\infty + \int_1^\infty s x^{-s-1} M(x,t)\,dx \\ &= \int_1^\infty s x^{-s-1} M(x,t)\,dx\end{aligned}$$

we then differentiate with respect to time and evaluate at $t = 0$

$$\begin{aligned}\varphi_t(s,0) &= \int_1^\infty s x^{-s-1} M_t(x,0)dx \\ &= \int_1^\infty s x^{-s-1}\left[-\phi\sum_{k=1}^{\infty} x^{-\beta\alpha^{-k}}\right]dx \\ &= -\phi\sum_{k=1}^{\infty}\frac{s}{\beta\alpha^{-k}+s}\end{aligned}$$

Finally, we use the fact the Mellin transform of a product of independent random variables is the product of their transforms. Therefore

$$\begin{aligned}\log\varphi(s,t) &= \lim_{n\to\infty} n\log\varphi\left(s,\frac{t}{n}\right) \\ &= t\lim_{n\to\infty}\frac{n}{t}\log\varphi\left(s,\frac{t}{n}\right) \\ &= t\lim_{\Delta\to 0}\frac{\log\varphi(s,\Delta)}{\Delta} \\ &= t\varphi_t(s,0) \\ &= t\left[-\phi\sum_{k=1}^{\infty}\frac{s}{\beta\alpha^{-k}+s}\right]\end{aligned}$$

Exponentiating both sides gives the result. ■

Claim 10 *Let $X_j(t)$ the random variable corresponding to firm j's proportional cost reduction in a period of length t. As t grows large,* $\frac{\log X_j(t) - \frac{\alpha}{1-\alpha}\frac{\phi}{\beta}t}{\sqrt{2\frac{\alpha^2}{1-\alpha^2}\frac{\phi}{\beta^2}t}}$ *converges in distribution to a standard normal random variable.*

Proof. This is just the central limit theorem. Let $\mu \equiv \frac{\alpha}{1-\alpha}\frac{\phi}{\beta}$ and $v \equiv \frac{\alpha^2}{1-\alpha^2}\frac{2\phi}{\beta^2}$, and let $y_j(t) =$

$\frac{\log X_j(t)-\mu t}{\sqrt{vt}}$. Using the Mellin transform of $X_j(t)$, the Laplace transform of y is

$$\begin{aligned}
E\left[e^{-y_j(t)s}\right] &= E\left[\exp\left\{-\left[\frac{\log X_j(t)-\mu t}{\sqrt{vt}}\right]s\right\}\right] \\
&= \int \exp\left\{-\left[\frac{\log x-\mu t}{\sqrt{vt}}\right]s\right\} M_x(x,t)\,dx \\
&= e^{\mu t\frac{s}{\sqrt{vt}}}\int x^{-\frac{s}{\sqrt{vt}}} M_x(x,t)\,dx \\
&= e^{\mu t\frac{s}{\sqrt{vt}}}\varphi\left(\frac{s}{\sqrt{vt}},t\right) \\
&= \exp\left\{\frac{s}{\sqrt{vt}}\mu t-\phi t\sum_{k=1}^{\infty}\frac{s/\sqrt{vt}}{\beta\alpha^{-k}+s/\sqrt{vt}}\right\}
\end{aligned}$$

Using $\mu=\frac{\alpha}{1-\alpha}\frac{\phi}{\beta}=\phi\sum_{k=1}^{\infty}\frac{1}{\beta\alpha^{-k}}$, this is

$$\begin{aligned}
E\left[e^{-y_j(t)s}\right] &= \exp\left\{\frac{s}{\sqrt{vt}}\phi t\sum_{k=1}^{\infty}\frac{1}{\beta\alpha^{-k}}-\phi t\sum_{k=1}^{\infty}\frac{s/\sqrt{vt}}{\beta\alpha^{-k}+s/\sqrt{vt}}\right\} \\
&= \exp\left\{\frac{s}{\sqrt{vt}}\phi t\sum_{k=1}^{\infty}\left(\frac{1}{\beta\alpha^{-k}}-\frac{1}{\beta\alpha^{-k}+s/\sqrt{vt}}\right)\right\} \\
&= \exp\left\{\frac{s}{\sqrt{vt}}\phi t\sum_{k=1}^{\infty}\frac{s/\sqrt{vt}}{\beta\alpha^{-k}\left(\beta\alpha^{-k}+s/\sqrt{vt}\right)}\right\} \\
&= \exp\left\{s^2\frac{\phi}{v}\sum_{k=1}^{\infty}\frac{1}{\beta\alpha^{-k}\left(\beta\alpha^{-k}+s/\sqrt{vt}\right)}\right\}
\end{aligned}$$

In the limit as t grows large is

$$\lim_{t\to\infty}\left(\frac{\phi}{v}\sum_{k=1}^{\infty}\frac{1}{\left(\beta\alpha^{-k}+s/\sqrt{vt}\right)\beta\alpha^{-k}}\right)=\frac{\phi}{v}\sum_{k=1}^{\infty}\frac{1}{\beta\alpha^{-k}\beta\alpha^{-k}}=\frac{1}{2}$$

so that

$$\lim_{t\to\infty}E\left[e^{-y_j(t)s}\right]=e^{-\frac{s^2}{2}}$$

which is the Laplace transform of a standard normal. ■

Let γ be the growth rate of the measure of entrants. Let $F(c)$ be the fraction of firms with cost no greater than c. Suppose that the distribution of cost among new firms has CDF F_0. Let φ^F and φ_0^F be the respective Mellin transforms.

Claim 11 *The Mellin transform of cost among entrants is*

$$\varphi_0^F(s)=\kappa_0^{\frac{\alpha}{\beta}s}\varphi^F(\beta)^{\frac{\alpha}{\beta}s}\Gamma\left(1-\frac{\alpha}{\beta}s\right)$$

Proof. A new entrant gets many initial draws of techniques. For each technique, the supplier and the match-specific productivity is random. The number of draws of techniques with match-specific component larger than z is $\kappa_0 z^{-\beta}$. Given z, the probability that the supplier's cost is low enough to deliver a cost lower than c is $\Pr\left(\left(\frac{c_s}{z_{bs}}\right)^{\alpha} \leq c|z\right) = \Pr\left(c_s \leq zc^{1/\alpha}|z\right) = F\left(zc^{1/\alpha}\right)$. As a result, the arrival rate of a draw that delivers effective cost smaller than c is $\kappa_0 \int_0^\infty F\left(zc^{1/\alpha}\right)\beta z^{-\beta-1}dz$. $1 - F^0(c)$ is the probability that no such draw arrives, which is thus

$$\begin{aligned}
1 - F^0(c) &= \exp\left\{-\kappa_0 \int_0^\infty F\left(zc^{1/\alpha}\right)\beta z^{-\beta-1}dz\right\} \\
&= \exp\left\{-\kappa_0 c^{\beta/\alpha} \int_0^\infty F(u)\beta u^{-\beta-1}du\right\}
\end{aligned}$$

using the change of variables $u = zc^{1/\alpha}$. Since $\int_0^\infty F(u)\beta u^{-\beta-1}du = \int_0^\infty u^{-\beta}dF(u) = \varphi^F(\beta)$, this is

$$F^0(c) = 1 - \exp\left\{-\kappa_0 \varphi^F(\beta) c^{\beta/\alpha}\right\}$$

The Mellin transform of initial cost is then

$$\begin{aligned}
\varphi_0^F(s) &\equiv \int_0^\infty c^{-s}dF^0(c) \\
&= \int_0^\infty c^{-s}\frac{\beta}{\alpha}\kappa_0\varphi^F(\beta) c^{\frac{\beta}{\alpha}-1}e^{-\kappa_0\varphi^F(\beta)\left(\frac{c}{w}\right)^{\frac{\beta}{\alpha}}}dc \\
&= \kappa_0^{\frac{\alpha}{\beta}s}\varphi^F(\beta)^{\frac{\alpha}{\beta}s}\int_0^\infty u^{-\frac{\alpha}{\beta}s}e^{-u}du
\end{aligned}$$

■

Claim 12 *The Mellin transform of F is*

$$\varphi^F(s) = \frac{1}{1 + \frac{\phi}{\gamma}\sum_{k=1}^\infty \frac{s}{s-\beta\alpha^{-k}}}\varphi_0^F(s)$$

Proof. For a firm of age τ, Each firms cost is the ratio of the cost they were born, c_{j0}, with and the proportional decline in cost since birth, x_j, $c_{jt} = \frac{c_{j0}}{x_j}$. Since these are independent, the Mellin transform of current cost among firms at age τ is

$$\varphi_\tau^F(s) = E\left[c_{jt}^{-s}\right] = E\left[c_{j0}^{-s}x_0^s\right] = E\left[c_{j0}^{-s}\right]E\left[x_0^s\right] = \varphi_0^F(s)\varphi^M(-s,\tau)$$

The Mellin transform of F is

$$\varphi^F(s) = \int_0^\infty \gamma e^{-\gamma\tau}\varphi_\tau^F(s,\tau)d\tau = \int_0^\infty \gamma e^{-\gamma\tau}\varphi_0^F(s)\varphi^M(-s,\tau)d\tau = \varphi_0^F(s)\int_0^\infty \gamma e^{-\gamma\tau}\varphi^M(-s,\tau)d\tau$$

Using the functional form for φ^M gives

$$\begin{aligned}\varphi^F(s) &= \varphi_0^F(s)\int_0^\infty \gamma e^{-\gamma\tau} e^{-\phi\tau\sum_{k=1}^\infty \frac{(-s)}{\beta\alpha^{-k}+(-s)}} d\tau \\ &= \varphi_0^F(s)\frac{\gamma}{\gamma+\phi\sum_{k=1}^\infty \frac{s}{s-\beta\alpha^{-k}}}\end{aligned}$$

■

Claim 13 *Let ν be the unique solution to $\frac{\gamma}{\phi} = \sum_{k=1}^\infty \frac{\nu}{\beta\alpha^{-k}-\nu}$. The distribution of cost in the cross-section decays has a power law left tail with exponent ν:*

$$\lim_{c\to 0}\frac{\log F(c)}{\log c} = \nu$$

Proof. First, note that $\varphi_0^F(s)$ is finite for all $s < \frac{\beta}{\alpha}$. We next show that $\frac{\gamma}{\phi} = \sum_{k=1}^\infty \frac{\nu}{\beta\alpha^{-k}-\nu}$ has a unique solution. $\sum_{k=1}^\infty \frac{\alpha^k}{\frac{\beta}{v}-\alpha^k}$ is continuous and strictly increasing in $\nu \in [0, \frac{\beta}{\alpha})$, taking the value of 0 for $\nu = 0$, diverging for $\nu \geq \frac{\beta}{\alpha}$, and negative for $\nu < 0$. There is therefore a unique value of ν such that $\sum_{k=1}^\infty \frac{\alpha^k}{\frac{\beta}{\nu}-\alpha^k} = \frac{\gamma}{\phi}$. ■

Next, the Mellin transform of the inverse of cost in the cross-section is $\varphi^F(-s)$. Thus this is also the Laplace transform of $y = -\log c$. Let H be the CDF of y, and let φ^H be its Laplace transform, so that $\varphi^H(s) = \varphi^F(-s)$. To characterize the tail behavior of of y, we use Theorem 3 of Nakagawa (2007) which states that if $-\nu$ is the abcissa of convergence[30] of the Laplace transform and a pole, then

$$\lim_{y\to\infty}\frac{\log(1-H(y))}{y} = -\nu \tag{11}$$

. Note that ν is such that $\frac{\phi}{\gamma}\sum_{k=1}^\infty \frac{\alpha^k}{\frac{\beta}{\nu}-\alpha^k} = 1$, and is $(-s) \gtrless v$ implies $\frac{\phi}{\gamma}\sum_{k=1}^\infty \frac{\alpha^k}{\frac{\beta}{(-s)}-\alpha^k} \gtrless 1$. Therefore if $s > -\nu$, $\varphi^H(s) = \frac{\varphi_F^0(-s)}{1-\frac{\phi}{\gamma}\sum_{k=1}^\infty \frac{\alpha^k}{\frac{\beta}{(-s)}-\alpha^k}}$ is finite because the denominator is positive and $\varphi_0^F(-s)$ is finite. But $\varphi^H(s)$ is negative (i.e., diverges) when $s < -\nu$. Further, $s = -\nu$ is a pole. Therefore we have the conclusion that (11) holds. We can use this along with $1 - H(y) = F(e^{-y})$ to get the left tail behavior of the distribution of cost:

$$\lim_{c\to 0}\frac{\log F(c)}{\log c} = -\lim_{y\to\infty}\frac{\log F(e^{-y})}{y} = -\lim_{y\to\infty}\frac{\log 1 - H(y)}{y} = \nu$$

Claim 14 *The Mellin transform of the cross-sectional distribution of cost is*

$$\varphi^F(s) = \frac{\Gamma\left(1-\frac{\alpha}{\beta}s\right)}{1+\frac{\phi}{\gamma}\sum_{k=1}^\infty \frac{s}{s-\beta\alpha^{-k}}}\left[\frac{\kappa_0\Gamma(1-\alpha)}{1+\frac{\phi}{\gamma}\sum_{k=1}^\infty \frac{1}{1-\alpha^{-k}}}\right]^{\frac{\alpha}{1-\alpha}\frac{s}{\beta}}$$

[30] An Abcissa of convergence of a Laplace transform $\mathcal{L}(s)$ is a negative number $\sigma_0 < 0$ such that $\mathcal{L}(s)$ diverges for $s < \sigma_0$ and converges for $s > \sigma_0$.

Proof. Evaluating the Mellin transforms of F and F^0 at β gives

$$\begin{aligned} \varphi^F(\beta) &= \frac{1}{1+\frac{\phi}{\gamma}\sum_{k=1}^{\infty}\frac{1}{1-\alpha^{-k}}}\varphi_0^F(\beta) \\ \varphi_0^F(\beta) &= \kappa_0^{\alpha}\varphi^F(\beta)^{\alpha}\Gamma(1-\alpha) \end{aligned}$$

combining these yields

$$\varphi^F(\beta) = \left[\frac{\kappa_0^{\alpha}\Gamma(1-\alpha)}{1+\frac{\phi}{\gamma}\sum_{k=1}^{\infty}\frac{1}{1-\alpha^{-k}}}\right]^{\frac{1}{1-\alpha}}$$

As a result, $\varphi_0^F(s)$ is

$$\varphi_0^F(s) = \left[\frac{\kappa_0\Gamma(1-\alpha)}{1+\frac{\phi}{\gamma}\sum_{k=1}^{\infty}\frac{1}{1-\alpha^{-k}}}\right]^{\frac{\alpha}{1-\alpha}\frac{s}{\beta}}\Gamma\left(1-\frac{\alpha}{\beta}s\right)$$

Pluggin this into the expression for $\varphi^F(s)$ gives the result. ■

In the case where $\varepsilon - 1 = \beta$, aggregate productivity is

$$\left[J_t\varphi^F(\varepsilon-1)\right]^{\frac{1}{\varepsilon-1}} = \left[J_t\varphi^F(\beta)\right]^{\frac{1}{\beta}} = J_t^{\frac{1}{\beta}}\left[\frac{\kappa_0^{\alpha}\Gamma(1-\alpha)}{1+\frac{\phi}{\gamma}\sum_{k=1}^{\infty}\frac{1}{1-\alpha^{-k}}}\right]^{\frac{1}{1-\alpha}\frac{1}{\beta}}$$

D.3 Full Model

For any firm in ω, let $M_\omega(x,t)$ be the probability that the firms cost declines by a proportion less than x. For a firm that uses industry ω as an input, let $K_\omega(x)$ be the probability that the firm's effective cost of using input ω declines by a factor less than x.

Suppose that new suppliers that deliver a proportional declines in cost larger than b arrive at rate $\phi_\omega b^{-\beta}$. What does this imply for the efficiency and aggregate productivity?

Let $M_\omega(x,t)$ be the probability that a firm producing product ω experiences a proportional decline in cost that is weakly less than x over an interval of length t.

Let $K_\omega(x,t)$ be the probability that a firm's effective cost of using industry ω declines by a factor weakly less than x over an interval of length t.

Define $m_\omega(x) \equiv \lim_{t\to 0}\frac{\partial M_\omega(x,t)}{\partial t}$ and $k_\omega(x) = \lim_{t\to 0}\frac{\partial K_\omega(x,t)}{\partial t}$

Claim 15 *m_ω and k_ω satisfy*

$$k_\omega(x) = m_\omega(x) - \phi_\omega x^{-\beta_\omega}$$

Proof. (Heuristic) A firm's effective cost of an input falls if its supplier's cost declines or if it switches to a different supplier. For a short enough time period t, there are two events that can happen to reduce a firm's effective cost of input ω by more than a factor x: either the firm finds a new supplier that delivers a jump in efficiency larger than x, or the existing supplier's unit cost declines by a factor of x (the full proof shows that the probability of some mixture of the two events is negligible). $K_\omega(x,t)$ is the probability that neither even happens:

$$\begin{aligned} K_\omega(x,t) &= M_\omega(x,t)\left\{e^{-t\phi_\omega \int_x^\infty \beta b^{-\beta-1}db}\right\} \\ &= M_\omega(x,t)e^{-t\phi_\omega x^{-\beta}} \end{aligned}$$

Differentiating with respect to t gives

$$\frac{\partial K_\omega(x,t)}{\partial t} = \frac{\partial M_\omega(x,t)}{\partial t} e^{-t\phi_\omega x^{-\beta}} - \phi_\omega x^{-\beta} K_\omega(x,t) e^{-t\phi_\omega x^{-\beta}}$$

taking the limit as $t \to 0$ gives

$$k_\omega(x) = m_\omega(x) - \phi_\omega x^{-\beta}$$

■

Proof. (Full) What is the probability that a firm's effective cost of ω declines by weakly less than the proportion x in an interval of length t? This could happen if either the firm does not find a new supplier and the existing supplier's cost falls by a factor weakly smaller than x, an event which occurs with probability $e^{-\phi_\omega t}M_\omega(x,t)$, or if the firm finds a new supplier that delivers a decline in cost with increment x_1 at some time $\tau \in [0,t]$, which occurs with density $e^{-\phi_\omega \tau}\phi_\omega \beta x_1^{-\beta-1}$, before which the original supplier's cost falls by a factor x_0, and after which effective cost improves by x_2, and $x_0 x_1 x_2 \leq x$. Thus

$$K_\omega(x,t) = e^{-\phi_\omega t}M_\omega(x,t) + \int_0^t \phi_\omega e^{-\phi_\omega \tau} \int_1^x \left\{ \int_1^{x/x_1} \left[K_\omega\left(\frac{x}{x_0 x_1}, t-\tau\right)\right] M_\omega(dx_0,\tau)\right\} \beta x_1^{-\beta-1} dx_1 d\tau$$

Taking the derivative with respect to t gives

$$\begin{aligned} \frac{\partial K_\omega(x,t)}{\partial t} &= -\phi e^{-\phi_\omega t} M_\omega(x,t) + e^{-\phi_\omega t}\frac{\partial M_\omega(x,t)}{\partial t} \\ &\quad + \phi_\omega e^{-\phi_\omega t} \int_1^x \left\{ \int_1^{x/x_1} \left[K_\omega\left(\frac{x}{x_0 x_1}, 0\right)\right] M_\omega(dx_0,t)\right\} \beta x_1^{-\beta-1} dx_1 \\ &\quad + \int_0^t \frac{d}{dt}\left\{ \phi_\omega e^{-\phi_\omega \tau} \int_1^x \left\{ \int_1^{x/x_1} \left[K_\omega\left(\frac{x}{x_0 x_1}, t-\tau\right)\right] M_\omega(dx_0,t)\right\} \beta x_1^{-\beta-1} dx_1 \right\} d\tau \end{aligned}$$

Note that $M(x,0) = K(x,0) = 1, \forall x \geq 1$. One implication is that the integral $\int_1^{x/x_1} K_\omega\left(\frac{x}{x_0 x_1}, 0\right) M_\omega(dx_0,t) = M_\omega\left(\frac{x}{x_1}, t\right)$. Taking the the limit as $t \to 0$ gives

$$\begin{aligned} k_\omega(x) &= -\phi_\omega + m_\omega(x) + \phi_\omega \int_1^x \beta x_1^{-\beta-1} dx_1 \\ &= -\phi_\omega + m_\omega(x) + \phi_\omega\left[-x^{-\beta}+1\right] \\ &= m_\omega(x) - \phi x^{-\beta} \end{aligned}$$

■

Let $\varphi_{\omega}^{M}(s,t) \equiv \int_1^\infty x^{-s} M_\omega(dx,t)$ and $\varphi_{\omega}^{K}(s,t) \equiv \int_1^\infty x^{-s} K_\omega(dx,t)$ be the Mellin transforms of $M_\omega(\cdot,t)$ and $K_\omega(\cdot,t)$ respectively. Since the Mellin transform of the product of random variables is the product of their respective transforms, these are related by

$$\varphi_{\hat{\omega}}^{K}(s,t) = \prod_{\omega} \varphi_{\omega}^{M}(\alpha_{\hat{\omega}\omega} s, t)$$

Similarly, define $\varphi_{\omega}^{m}(s) \equiv \lim_{t\to 0} \frac{\varphi_{\omega}^{M}(s,t)-1}{t}$ and $\varphi_{\omega}^{k}(s) \equiv \lim_{t\to 0} \frac{\varphi_{\omega}^{K}(s,t)-1}{t}$.

Lemma 7 *The Mellin transforms of satisfy $\varphi_{\omega}^{K}(s,t) = e^{t\varphi_{\omega}^{k}(s)}$ and $\varphi_{\omega}^{M}(s,t) = e^{t\varphi_{\omega}^{m}(s)}$. In addition, $\varphi_{\omega}^{m}(s) = \int_1^\infty s x^{-s-1} m_\omega(x)dx$ and $\varphi_{\omega}^{k}(s) = \int_1^\infty s x^{-s-1} k_\omega(x)dx$*

Proof. We provide the proof for φ_{ω}^{M}; the proof for φ_{ω}^{K} is identical. Integrating by parts, φ_{ω}^{M} can be expressed as

$$\begin{aligned}\varphi_{\omega}^{M}(s,t) &= \int_1^\infty x^{-s} M_\omega(dx,t) = x^{-s} M_\omega(x,t)\big|_1^\infty + \int s x^{-s-1} M_\omega(x,t)dx \\ &= \int_1^\infty s x^{-s-1} M_\omega(x,t)dx\end{aligned}$$

Next, taking limits and using L'Hospital's rule, φ_{ω}^{m} and m are related to each other by

$$\varphi_{\omega}^{m}(s) = \lim_{t\to 0} \frac{\varphi_{\omega}^{M}(s,t)-1}{t} = \lim_{t\to 0} \frac{d}{dt}\left[\varphi_{\omega}^{M}(s,t)\right] = \lim_{t\to 0} \frac{d}{dt}\left[\int_1^\infty s x^{-s-1} M_\omega(x,t)dx\right] = \int_1^\infty s x^{-s-1} m_\omega(x)dx$$

To see that $\varphi_{\omega}^{M}(s,t) = e^{t\varphi_{\omega}^{m}(s)}$, note that since the growth in each subperiod is independent, we have

$$\begin{aligned}\frac{\partial \log \varphi_{\omega}^{M}(s,t)}{\partial t} &= \frac{1}{\varphi_{\omega}^{M}(s,t)} \frac{\partial \varphi_{\omega}^{M}(s,t)}{\partial t} = \frac{1}{\varphi_{\omega}^{M}(s,t)} \lim_{\Delta\to 0} \frac{\varphi_{\omega}^{M}(s,t+\Delta) - \varphi_{\omega}^{M}(s,t)}{\Delta} \\ &= \frac{1}{\varphi_{\omega}^{M}(s,t)} \lim_{\Delta\to 0} \frac{\varphi_{\omega}^{M}(s,t)\varphi_{\omega}^{M}(s,\Delta) - \varphi_{\omega}^{M}(s,t)}{\Delta} = \lim_{\Delta\to 0} \frac{\varphi_{\omega}^{M}(s,\Delta) - 1}{\Delta} = \varphi_{\omega}^{m}(s)\end{aligned}$$

Liebniz rule then implies

$$\log \varphi_{\omega}^{M}(s,t) = \int_0^t \frac{d \log \varphi_{\omega}^{M}(s,\tau)}{d\tau} d\tau = \int_0^t \varphi_{\omega}^{m}(s) d\tau = t\varphi_{\omega}^{m}(s) .$$

■

Claim 16 *φ_{ω}^{k} and φ_{ω}^{m} satisfy the following two relationships*

$$\varphi_{\hat{\omega}}^{m}(s) = \sum_{\omega} \varphi_{\omega}^{k}(\alpha_{\hat{\omega}\omega} s)$$

$$\varphi_{\omega}^{k}(s) = \varphi_{\omega}^{m}(s) - \frac{\phi_\omega s}{\beta + s}$$

Proof. For the first relationship, we can use $\varphi^M_{\hat{\omega}}(s,t) = \prod_\omega \varphi^K_\omega(\alpha_{\hat{\omega}\omega} s, t)$ to express $\varphi^m_{\hat{\omega}}(s)$ as

$$\begin{aligned}
\varphi^m_{\hat{\omega}}(s) &= \lim_{t \to 0} \frac{\varphi^M_{\hat{\omega}}(s,t) - 1}{t} \\
&= \lim_{t \to 0} \frac{\prod_\omega \varphi^K_\omega(\alpha_{\hat{\omega}\omega} s, t) - 1}{t} \\
&= \lim_{t \to 0} \frac{d}{dt} \left\{ \prod_\omega \varphi^K_\omega(\alpha_{\hat{\omega}\omega} s, t) \right\} \\
&= \lim_{t \to 0} \varphi^M_{\hat{\omega}}(s,t) \sum_\omega \frac{\partial \log \varphi^K_\omega(\alpha_{\hat{\omega}\omega} s, t)}{\partial t} \\
&= \sum_\omega \varphi^k_\omega(\alpha_{\hat{\omega}\omega} s)
\end{aligned}$$

where the last line used $\lim_{t\to 0} \varphi^M_{\hat{\omega}}(s,t) = 1$ and $\lim_{t \to 0} \frac{\partial \log \varphi^K_\omega(\alpha_{\hat{\omega}\omega} s,t)}{\partial t} = \varphi^k_\omega(\alpha_{\hat{\omega}\omega} s)$.

For the second relationship, we can use $k_\omega(x) = m_\omega(x) - \phi_\omega x^{-\beta}$ to express $\varphi^k_\omega(s)$ as

$$\begin{aligned}
\varphi^k_\omega(s) &= \int_1^\infty s x^{-s-1} k_\omega(x) dx \\
&= \int_1^\infty s x^{-s-1} \left[m_\omega(x) - \phi_\omega x^{-\beta} \right] dx \\
&= \varphi^m_\omega(s) - \phi_\omega s \int_1^\infty x^{-s-1} x^{-\beta} dx \\
&= \varphi^m_\omega(s) - \frac{\phi_\omega s}{\beta + s}
\end{aligned}$$

■

We next derive an explicit (rather than recursive) expression for these transforms

Claim 17 *Let $J^m(s)$ be the vector with representative element $\varphi^m_{\hat{\omega}}(s)$. Then*

$$J^m(s) = \left\{ \sum_{n=1}^\infty (-s/\beta)^n (I - \alpha_n)^{-1} \alpha_n \right\} \Phi$$

where α_n is the $\Omega \times \Omega$ matrix with typical element $(\alpha_n)_{\hat{\omega}\omega} = \alpha^n_{\hat{\omega}\omega}$ and Φ is vector with elements ϕ_ω.

Proof. We will do a Taylor expansion of $\varphi^m_{\hat{\omega}}(s)$ around $s = 0$. From above, we have that

$$\begin{aligned}
\varphi^m_{\hat{\omega}}(s) &= \sum_\omega \varphi^m_\omega(\alpha_{\hat{\omega}\omega} s) - \frac{\phi_\omega \alpha_{\hat{\omega}\omega} s}{\beta + \alpha_{\hat{\omega}\omega} s} \\
&= \sum_\omega \varphi^m_\omega(\alpha_{\hat{\omega}\omega} s) - \phi_\omega + \frac{\phi_\omega \beta}{\beta + \alpha_{\hat{\omega}\omega} s}
\end{aligned}$$

Noting that $\frac{d^n}{dx^n}\left(\frac{1}{\beta+\alpha x}\right) = (-\alpha)^n n! \frac{1}{(\beta+\alpha x)^{n+1}}$ the n^{th} derivative of this equation (for $n \geq 1$) is

$$\varphi_{\hat{\omega}}^{m(n)}(s) = \sum_{\omega} \alpha_{\hat{\omega}\omega}^{n} \varphi_{\omega}^{m(n)}(\alpha_{\hat{\omega}\omega} s) + \phi_{\omega} \beta \left(-\alpha_{\hat{\omega}\omega}\right)^{n} n! \frac{1}{(\beta + \alpha_{\hat{\omega}\omega} s)^{n+1}}$$

or, evaluating at $s = 0$,

$$\varphi_{\hat{\omega}}^{m(n)}(0) = \sum_{\omega} \alpha_{\hat{\omega}\omega}^{n} \varphi_{\omega}^{m(n)}(0) + \phi_{\omega} \left(-\frac{\alpha_{\hat{\omega}\omega}}{\beta}\right)^{n} n!$$

In vector form, this is

$$\begin{aligned} J^{m(n)}(s) &= \alpha_n J^{m(n)}(s) + \alpha_n (-\beta)^{-n} \Phi n! \\ J^{m(n)}(s) &= (-\beta)^{-n} \left(I - \alpha_n\right)^{-1} \alpha_n \Phi n! \end{aligned}$$

Using $\varphi_{\hat{\omega}}^{m}(0) = 0$, the Taylor expansion around $s = 0$ is

$$\begin{aligned} J^{m}(s) &= \sum_{n=0}^{\infty} \frac{s^n J^{m(n)}(0)}{n!} \\ &= \left\{ \sum_{n=1}^{\infty} (-s/\beta)^{n} \left(I - \alpha_n\right)^{-1} \alpha_n \right\} \Phi \end{aligned}$$

■

Claim 18 *For firm j in industry ω, let $X_j(t)$ the random variable corresponding to the firm's proportional cost reduction in a period of length t. As t grows large, $\frac{\log X_j(t) - \mu_\omega}{\sqrt{v_\omega t}}$ converges in distribution to a standard normal random variable, where $\mu_\omega \equiv -\varphi_\omega^{m\prime}(0)$ and $v_\omega \equiv \varphi_\omega^{m\prime\prime}(0)$ satisfy*

$$\begin{aligned} \mu_\omega &= \sum_{\hat{\omega}} \alpha_{\omega\hat{\omega}} \left(\mu_{\hat{\omega}} + \frac{\phi_{\hat{\omega}}}{\beta}\right) \\ v_\omega &= \sum_{\hat{\omega}} \alpha_{\omega\hat{\omega}}^{2} \left(v_{\hat{\omega}} + \frac{\phi_{\hat{\omega}}}{\beta^2}\right) \end{aligned}$$

Proof. Let $Y_j(t) = \frac{\log X_j(t) - \mu_\omega t}{\sqrt{v_\omega t}}$. The Mellin transform of $X_j(t)$ is $E\left[X_j(t)^{-s}\right] = \varphi_\omega^{M}(s,t) = e^{t\varphi_\omega^{m}(s)}$. The Laplace transform of $Y_j(t)$ is

$$E\left[e^{-Y_j(t)s}\right] = E\left[e^{-\left[\frac{\log X_j(t) - \mu_\omega t}{\sqrt{v_\omega t}}\right]s}\right] = e^{\frac{\mu_\omega t}{\sqrt{v_\omega t}} s} E\left[X_j(t)^{-\frac{s}{\sqrt{v_\omega t}}}\right]$$

Using $E\left[X_j(t)^{-\frac{s}{\sqrt{v_\omega t}}}\right] = \varphi_\omega^{M}(\frac{s}{\sqrt{v_\omega t}}) = \exp \varphi_\omega^{m}\left(\frac{s}{\sqrt{v_\omega t}}\right)$, the Laplace transform of $Y_j(t)$ is then

$$E\left[e^{-Y_j(t)s}\right] = e^{\left[\varphi_\omega^{m}\left(\frac{s}{\sqrt{v_\omega t}}\right) + \mu_\omega \frac{s}{\sqrt{v_\omega t}}\right]t}$$

We can take the limit of the exponent. Using $\mu_\omega = -\varphi_\omega^{m\prime}(s)$, the change of variables $u = \frac{1}{\sqrt{v_\omega t}}$, using L'Hopital's rule twice, and $v_\omega \equiv \varphi_\omega^{m\prime\prime}(0)$, the exponent is:

$$\begin{aligned}
\lim_{t\to\infty} t\left[\varphi_\omega^m\left(\frac{s}{\sqrt{v_\omega t}}\right) - \varphi_\omega^{m\prime}(0)\frac{s}{\sqrt{v_\omega t}}\right] &= \lim_{u\to 0}\frac{\varphi_\omega^m(us) - \varphi_\omega^{m\prime}(0)su}{v_\omega u^2} \\
&= \lim_{u\to 0}\frac{s\varphi_\omega^{m\prime}(us) - \varphi_\omega^{m\prime}(0)s}{v_\omega 2u} \\
&= \lim_{u\to 0}\frac{s^2\varphi_\omega^{m\prime\prime}(0)}{v_\omega 2} \\
&= \frac{s^2}{2}
\end{aligned}$$

so that

$$\lim_{t\to\infty} E\left[e^{-y_j(t)s}\right] = e^{-\frac{s^2}{2}}$$

which is the Laplace transform of a standard normal.

Finally, we derive expressions for μ_ω and v_ω. Starting from the expression for $\varphi_\omega^m(s)$,

$$\varphi_\omega^m(s) = \sum_{\hat{\omega}} \varphi_{\hat{\omega}}^m\left(\alpha_{\omega\hat{\omega}}s\right) - \frac{\phi_\omega\alpha_{\omega\hat{\omega}}s}{\beta + \alpha_{\omega\hat{\omega}}s}$$

Differentiating once, evaluating at $s = 0$ and using $\mu_\omega = -\varphi_\omega^{m\prime}(0)$ gives

$$\mu_\omega = \sum_{\hat{\omega}} \alpha_{\omega\hat{\omega}}\left(\mu_{\hat{\omega}} + \frac{\phi_{\hat{\omega}}}{\beta}\right)$$

Differentiating twice, evaluating at $s = 0$ and using $v_\omega = \varphi_\omega^{m\prime\prime}(0)$ gives

$$v_\omega = \sum_{\hat{\omega}} \alpha_{\omega\hat{\omega}}^2\left(v_{\hat{\omega}} + \frac{\phi_{\hat{\omega}}}{\beta^2}\right)$$

■

We now study the distribution of cost in the cross section and among entrants. We assume that when a firm in ω is born, the number of potential suppliers in industry $\hat{\omega}$ with match-specific productivity larger than z is $\kappa_{0,\omega\hat{\omega}}z^{-\beta}$.

Let $F_\omega(c)$ be the cumulative distribution of cost among firms in ω. Let $F_{\omega,a}$ be the cumulative distribution of cost among firms of age a, so that $F_{\omega,0}$ is the distribution function among entrants. Let $\varphi_\omega^F(c) = \int c^{-s}dF_\omega(c)$ and $\varphi_{\omega,a}^F(c) = \int c^{-s}dF_{\omega,a}(c)$ be their respective Mellin transforms.

Proposition 8 *The Mellin transforms of the cross-sectional distribution of cost in ω and the distribution of cost among entrants in ω satisfy*

$$\varphi_\omega^F(s) = \frac{\gamma}{\gamma - \varphi_\omega^m(-s)}\varphi_{\omega,0}^F(s) \tag{12}$$

$$\varphi^F_{\omega,0}(s) = \prod_{\hat{\omega}} \left(\kappa_{0\omega\hat{\omega}} \varphi^F_{\hat{\omega}}(\beta)\right)^{\alpha_{\omega\hat{\omega}} s/\beta} \Gamma\left(1 - \frac{\alpha_{\omega\hat{\omega}} s}{\beta}\right) \quad (13)$$

Proof. The cross-sectional distribution of cost satisfies

$$F_\omega(c) = \int_0^\infty \gamma e^{-\gamma a} F_{\omega,a}(c) da$$

so that

$$\varphi^F_\omega(s) = \int_0^\infty \gamma e^{-\gamma a} \varphi^F_{\omega,a}(s) da$$

Since a firm's cost is the product of its cost at birth and the change in cost between birth and age a,

$$\varphi^F_{\omega,a}(s) = \varphi^F_{\omega,0}(s) \varphi^M_\omega(-s, a) = \varphi^F_{\omega,0}(s) e^{\varphi^m_\omega(-s)a}$$

Putting these together gives

$$\varphi^F_\omega(s) = \int_0^\infty \gamma e^{-\gamma a} \varphi^F_{\omega,0}(s) e^{\varphi^m_\omega(-s)a} da = \frac{\gamma}{\gamma - \varphi^m_\omega(-s)} \varphi^F_{\omega,0}(s)$$

To get at the distribution of cost at birth, we assume that when a firm in ω is born, the number of potential suppliers in industry $\hat{\omega}$ with match-specific productivity larger than z is $\kappa_{0,\omega\hat{\omega}} z^{-\beta}$. As a result, the arrival rate of a supplier of $\hat{\omega}$ that delivers effective cost lower than u is

$$\int_0^\infty F_{\hat{\omega}}(zu) \kappa_{0\omega\hat{\omega}} \beta z^{-\beta-1} dz = \kappa_{0\omega\hat{\omega}} u^\beta \int_0^\infty F_{\hat{\omega}}(a) \beta a^{-\beta-1} da = \kappa_{0\omega\hat{\omega}} u^\beta \int_0^\infty a^{-\beta} dF_{\hat{\omega}}(a) = \kappa_{0\omega\hat{\omega}} u^\beta \varphi^F_{\hat{\omega}}(\beta)$$

So the probability that an entrant's effective cost of $\hat{\omega}$ is lower than u is $1 - e^{-\kappa_{0\omega\hat{\omega}} u^\beta \varphi^F_{\hat{\omega}}(\beta)}$. The Mellin transform of the effective cost of input $\hat{\omega}$ at birth is thus

$$\begin{aligned}
E\left[u^{-s}\right] &= \int_0^\infty u^{-s} \kappa_{0\omega\hat{\omega}} \beta u^{\beta-1} \varphi^F_{\hat{\omega}}(\beta) e^{-\kappa_{0\omega\hat{\omega}} u^\beta \varphi^F_{\hat{\omega}}(\beta)} du \\
&= \left(\kappa_{0\omega\hat{\omega}} \varphi^F_{\hat{\omega}}(\beta)\right)^{s/\beta} \int_0^\infty x^{-s/\beta} e^{-x} dx \\
&= \left(\kappa_{0\omega\hat{\omega}} \varphi^F_{\hat{\omega}}(\beta)\right)^{s/\beta} \Gamma\left(1 - \frac{s}{\beta}\right)
\end{aligned}$$

Since cost at birth is the weighted product of the effective cost of the inputs, and these are independent,

$$\varphi^F_{\omega,0}(s) = \prod_{\hat{\omega}} \left(\kappa_{0\omega\hat{\omega}} \varphi^F_{\hat{\omega}}(\beta)\right)^{\alpha_{\omega\hat{\omega}} s/\beta} \Gamma\left(1 - \frac{\alpha_{\omega\hat{\omega}} s}{\beta}\right)$$

■

Proposition 9 *The cross-sectional distributions of cost satisfy*

$$\varphi^F_\omega(\beta) = \frac{\gamma}{\gamma - \varphi^m_\omega(-\beta)} \Xi_\omega \prod_{\hat{\omega}} \varphi^F_{\hat{\omega}}(\beta)^{\alpha_{\omega\hat{\omega}}}$$

where $\Xi_\omega \equiv \prod_{\hat{\omega}} \kappa_{0\omega\hat{\omega}}^{\alpha_{\omega\hat{\omega}}} \Gamma\left(1-\alpha_{\omega\hat{\omega}}\right)$.

Proof. Evaluating (13) at $s=\beta$ gives

$$\varphi_{\omega,0}^F(\beta) = \prod_{\hat{\omega}} \left(\kappa_{0\omega\hat{\omega}} \varphi_{\hat{\omega}}^F(\beta)\right)^{\alpha_{\omega\hat{\omega}}} \Gamma\left(1-\alpha_{\omega\hat{\omega}}\right) = \Xi_\omega \prod_{\hat{\omega}} \varphi_{\hat{\omega}}^F(\beta)^{\alpha_{\omega\hat{\omega}}}$$

Plugging this into (12) evaluated at $s=\beta$ gives the result. ∎